Acta Acustica united with Acustica
Vol. 92 (2006) 202 – 209

Coupled Equations for Particle Velocity and Temperature Variation as the Fundamental Formulation of Linear Acoustics in Thermo-Viscous Fluids at Rest

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Summary
This paper addresses a formulation of linear acoustics in thermo-viscous fluids at rest, in the form of a set of two equations for the temperature variation and the particle velocity. These variables are suitable to obtain explicit representation of the total perturbation field, because they are relevant to describe the scalar, potential acoustic and entropic movements, and the associated shear movement induced by viscosity effects. These variables are also relevant to derive boundary conditions, including both the thermal diffusion process through the interfaces between fluid and boundaries and the velocity field of these boundaries. This formulation offers analytical and numerical tools for solving most of the practical problems in thermo-viscous fluids (including thermo-viscous boundary layers) which was not available until now. In order to show the advantages and the efficiency of the formulation, a simple analytical solution is provided for the reflection of a plane wave, accounting for the viscous and thermal effects in the fluid and for thermal diffusion in the rigid wall boundary.

PACS no. 43.20.Hq, 43.20.Tb, 43.30.Es, 43.35.Ae

1. Introduction

In the framework of linear acoustics in thermo-viscous fluids at rest, including or not the coupling with vibrating structures, in infinite, bounded or closed domains, the analytical formulation and the numerical modelling currently used, when dealing with fundamental or engineering activities, lie on specific approximations for idealized problems modelling the real ones. Regarding analytic solutions, those available are appropriate only to addressing very specific examples, that is specialized geometries and idealizations, including those which can be solved involving appropriate complex wavenumbers in the bulk of the domain and specific admittances at the boundaries (which describes the local, reactive and dissipative properties of the wall materials). But in much of the more practical recent literature, works are extensively numerical and rely on standard generic finite-element programs: these available finite element or boundary element methods [1] [2] are based on potential acoustic movements, then, they are not compatible with both the modelling of reacting and absorbing processes due to shear viscosity effects and the induced vortical movements (which play a prominent role inside the boundary layers). Moreover, they do not take into account the thermal effects which play also an important role in many applications.

The motivation to solve new classes of problems in dissipative, thermoviscous fluids arises during the last decade from the requirement of explicit representation of the viscous and the thermal boundary layer effects, especially in cavities where the thermo-viscous boundary layers cause significant dissipation [3], or when the interactions between the actual acoustic movement, the entropic one (due to thermal diffusion) and the vortical one (due to shear viscosity) must be modelled accurately [4] [5] (this last approach must be considered in very small cavities and narrow ducts, where one or two dimensions are similar in magnitude to the boundary layer thicknesses). Despite recent theoretical works on the subject, which are appropriate only to addressing specific examples, a relevant global formulation satisfying these requirements is not yet available. In our knowledge, publications related to this subject include the papers by E. Dokumaci [2, 3], A. Cummings [4], W. M. Beltman *et al.* [5], C. Karra *et al.* [6], and R. Bossart *et al.* [7]. The works of E. Dokumaci (1991–1995) lead to both bi-dimensional modelling of viscous boundary layers of semi-infinite domains using a boundary element method [2] and the modelling of heat conduction effects [3]. The works of A. Cummings [4] and W. M. Beltman *et al.* [5] make use of numerical methods to model thin layers of viscous fluid trapped between

Received 12 May 2005,
accepted 22 October 2005.

parallel plane walls. C. Karra *et al.*, through a global approach [6], establish a tri-dimensional integral formulation for a non viscous, but heat conducting fluid (because the available packaging, based on potential acoustics, cannot model the vortical movements). R. Bossart *et al.* described more recently [7] a hybrid numerical and analytical procedure for calculating acoustic fields in thermoviscous fluids inside bounded domains, which are relevant only when all dimensions of the domains considered are much greater than the boundary layer thicknesses, the dissipation effect of the boundary layers on the acoustic field being modelled by a known impedance-like boundary condition.

The need for a relevant global formulation satisfying the practical requirements, providing both analytical and numerical solutions not only for the actual acoustic fields, but also for the associated entropic and vortical fields, clearly arises when dealing with practical classes of applications, namely porous materials, inertial acoustics, thermoacoustics, transducers, capillary domains, even infinite or semi infinite domains, closed domains, and so on. The present paper deals with such a fundamental formulation. A set of two coupled differential equations is derived for the temperature variation and the particle velocity, couple of suitable variables to describe both the scalar, potential movements (acoustic and entropic movements) and the vortical movement. These variables are moreover relevant to derive boundary conditions, including the thermal diffusion process through the interfaces and the tangential and normal movement of boundaries, the virtual interfaces between two domains filled with the same fluid being particular cases of those mentioned here (continuity equations for pressure variation, temperature variation, heat flux, particle velocity, and so on).

2. Total perturbation field in thermoviscous fluids at rest

The fluid considered is Stokesian (stress proportional to rate of strain and heat flux proportional to temperature gradient), homogeneous and at rest, with a linear behaviour for the differential form of its equation of state. Linearity of the total perturbation movement is the only assumption made in the formulation. Viscous and thermal phenomena induce vortical (shear) and entropic (potential) movements, giving rise to relaxation phenomena in the bulk of the fluid which cause dissipation of acoustic energy. On the boundaries, reactive and absorbing processes arise from interactions between the acoustic movement and both the entropic movement (diffusion of heat) and the vortical movement (diffusion of shear wave) which are created on the boundary wall, pumping energy from the acoustic wave inside the so-called viscous and thermal boundary layers (the amplitude of the shear waves and heat perturbations created at the boundary have the same order of magnitude than the acoustic wave itself as a consequence of the non slip condition and the quasi isothermal condition respectively). In what follows in this section, the mathematical model of the fluid is briefly recapitulated, in order that the presentation be self-contained. Then, the set of two basic equations of the general formulation mentioned before and the associated boundary conditions are presented.

A thermoviscous fluid oscillating around some steady state can be described by a set of thermodynamical variables, mainly the pressure variation p, the particle velocity $\boldsymbol{v}$, the density variation ρ', the entropy variation per unit mass s and the temperature variation τ, all of which being supposed to be small so that linear approximation remains valid. The thermostatic state and the nature of the fluid are then accounted for by thermostatic parameters and by phenomenological quantities respectively: the ambient values P_0 of the pressure, T_0 of the temperature, and ρ_0 of the density, and the adiabatic speed of sound $c_0 = \sqrt{\gamma/(\rho_0 \chi_T)}$, the shear viscosity μ, the bulk viscosity η, the coefficient of thermal conductivity λ, the heat coefficients at constant pressure and constant volume per unit of mass C_p and C_v, their ratio $\gamma = C_p/C_v$, the increase in pressure per unit increase in temperature at constant density $\hat{\beta} = (\partial P/\partial T)_\rho$ and the isothermal compressibility $\chi_T = 1/\rho(\partial\rho/\partial p)_T$. The fundamental laws of mechanics, taking into account the phenomenological complementary relationships of the fluid, take the following linearized form:

$i-$ the Stokes-Navier equation

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} + \mathbf{grad}\, p - \rho_0 c_0\, \ell_v\, \mathbf{grad} \operatorname{div} \boldsymbol{v} + \rho_0 c_0\, \ell'_v\, \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v} = \mathbf{0}, \tag{1}$$

where the viscous characteristic lengths ℓ_v and ℓ'_v are defined as

$$\ell_v = (\eta + \frac{4}{3}\mu)/\rho_0 c_0 \quad \text{and} \quad \ell'_v = \mu/\rho_0 c_0.$$

$ii-$ the conservation of mass equation

$$\frac{\partial \rho'}{\partial t} + \rho_0 \operatorname{div} \boldsymbol{v} = 0. \tag{2}$$

$iii-$ the conservation of energy, which reduces to

$$\rho_0\, T_0 \frac{\partial s}{\partial t} - \rho_0\, c_0\, C_p\, \ell_h \operatorname{div} \mathbf{grad}\, \tau = 0, \tag{3}$$

where the thermal characteristic length is defined as

$$\ell_h = \lambda/(\rho_0 c_0 C_p) = \xi/\rho_0,$$

$\xi = \lambda/(\rho_0 C_p)$ being the thermal diffusivity of the acoustic fluid.

The thermodynamical state laws of the fluid allow us to express all thermodynamical quantities with only two independent variables. Choosing to keep p and τ leads to the following state equations:

$$\rho' = \frac{\gamma}{c_0^2}(p - \hat{\beta}\tau), \tag{4}$$

$$s = \frac{C_p}{T_0}\Big(\tau - \frac{\gamma - 1}{\hat{\beta}\gamma} p\Big). \tag{5}$$

At the locations of sources, the right hand sides of equations (1), (2) and (3) do not vanish anymore [8]: they represent respectively the force sources, the volume velocity sources and the heat sources. But, in the remainder of the paper, equations are expressed always at all locations outside the sources (therefore avoiding complications which overshadow the purpose of the paper).

Apart from a slight temperature jump and a slight velocity slip at the boundary, both of which negligible in most of the real situations (except in rarefied gases), the most general boundary conditions, at the interfaces between fluids and hard boundaries, assume first the continuity of the temperature and the heat flux, and second the continuity of the particle velocity and the constraint vectors. These boundary conditions can be expressed by using two variables, the temperature (scalar variable) and the total particle velocity (vector variable). Moreover, both temperature variation τ and particle velocity $\boldsymbol{v}$ are suitable variables for describing the vortical and entropic movements induced by viscous and thermal phenomena in the bulk of the fluid. Then, combining equations (1), (2), (3), (4) and (5) to remove the variables p, ρ' and s leads to the set of two coupled equations involving the variables $\boldsymbol{v}$ and τ:

$$-\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \left(\frac{c_0^2}{\gamma} + c_0 \ell_v \frac{\partial}{\partial t}\right) \mathbf{grad}\,\mathrm{div}\,\boldsymbol{v} - c_0 \ell'_v \frac{\partial}{\partial t}\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} - \frac{\hat{\beta}}{\rho_0}\frac{\partial}{\partial t}\,\mathbf{grad}\,\tau = \mathbf{0}, \tag{6}$$

$$\frac{\partial \tau}{\partial t} - \gamma \ell_h c_0\,\mathrm{div}\,\mathbf{grad}\,\tau + \frac{\gamma - 1}{\gamma \hat{\beta}} \rho_0 c_0^2\,\mathrm{div}\,\boldsymbol{v} = 0, \tag{7}$$

and, invoking equations (1-5), the other variables can be expressed from the variables $\boldsymbol{v}$ and τ. For instance, the time-derivative of the total pressure perturbation is expressed as

$$\frac{\partial p}{\partial t} = \hat{\beta}\frac{\partial \tau}{\partial t} - \frac{\rho_0 c_0^2}{\gamma}\mathrm{div}\,\boldsymbol{v}. \tag{8}$$

The usual perturbation equation is obtained from the set (6)–(7) as the limit of non dissipative fluids : setting $\ell_v = \ell'_v = 0$ (unviscous fluid), $\ell_h = 0$ (no thermal diffusion) and injecting equation (7) into (6) gives the vectorial equation for particle velocity $\partial^2 \boldsymbol{v}/\partial t^2 - c_0^2\,\mathbf{grad}\,\mathrm{div}\,\boldsymbol{v} = \mathbf{0}$, (completed by $\mathbf{curl}\,\boldsymbol{v} = \mathbf{0}$ without shear viscous stress), giving from (7) and (8) the scalar equation $(\partial^2/\partial t^2 - c_0^2\,\mathrm{div}\,\mathbf{grad}\,) = 0$ for the temperature and pressure time-derivative.

Therefore, the total perturbation movement (that is the sum of the acoustic, entropic and vortical movements) is governed by this set of coupled equations (6) and (7). It is subject to the general boundary conditions for the temperature variation τ and the particle velocity $\boldsymbol{v}$, namely (the variables and the properties of the solid boundaries being denoted using the subscript "w"):

$i-$ the continuity of the temperature variation and the normal heat flux (thermal boundary conditions)

$$\tau = \tau_w, \tag{9}$$

$$\lambda\mathbf{grad}\,\tau \cdot \boldsymbol{n} = \left(\underline{\underline{\lambda_w}}\,\mathbf{grad}\,\tau_w\right) \cdot \boldsymbol{n}, \tag{10}$$

where $\underline{\underline{\lambda_w}}$ is the thermal conductivity tensor of the solid boundary,

$ii-$ the continuity of the velocity (that is of both the normal and tangential to the wall components of the particle velocity) and of the normal constraint vector (mechanical boundary conditions), *i.e.* the non slip condition

$$\boldsymbol{v} = \frac{\partial \boldsymbol{u}_w}{\partial t}, \tag{11}$$

$$\left(-p + (\eta - 2\mu/3)\,\mathrm{div}\,\boldsymbol{v}\right)\boldsymbol{n} + 2\mu\,\underline{\underline{v}}\,\boldsymbol{n} = \underline{\underline{\sigma_w}}\,\boldsymbol{n}, \tag{12}$$

where the pressure variation p is given by equations (2) and (4), $\underline{\underline{v}}$ is the symmetrical rate of strain tensor in the fluid $v_{ij} = 1/2[(\partial v_i/\partial x_j) + (\partial v_j/\partial x_i)]$, and $\underline{\underline{\sigma_w}}$ is the stress tensor in the solid, defined as $\sigma_{w_{ij}} = D_{ijkl}(\partial u_{wk}/\partial x_l)$, D_{ijkl} being the bulk elasticity coefficients of the boundary material and $\boldsymbol{u}_w$ being the displacement of the solid.

Except for time derivation, the operator acting on the particle velocity in equation (6) and the operator acting on the temperature variation in equation (7) have respectively the same structure as the classical elastodynamic and diffusion operators. Then, regarding harmonic solutions, this formulation can be numerically implemented for calculating the acoustic perturbation in thermoviscous fluids by analogy with current packagings used for elastodynamic and thermal diffusion problems.

It is noteworthy that in most cases, fluids are restricted to dense gases, and boundaries to rigid, massive and heat conducting boundaries ; the boundary conditions then reduce to the homogeneous Dirichlet conditions, namely $\tau = 0$ for the temperature variation (which represents the limiting isothermal case) and $\boldsymbol{v} = \boldsymbol{0}$ for the particle velocity (non slip condition).

3. Locally plane boundary: complete solution for impedance-like functions

As mentioned previously, the usefulness of the model presented in this paper is basically numerical and relies on standard numerical methods, like finite-element modelling. It is not the purpose of this work to test the numerical implementation against some standard benchmark analytic solutions. The remaining subject matter of the paper is to highlight the relevance of the formalism when the total acoustic perturbation phenomena, including viscous- and thermal-effects, must be modelled precisely to obtain accurate results. A simple example involving the plane wave reflection on a rigid plane boundary is considered in this section: the admittance-like function β of the boundary is obtained analytically from equations (6)–(12) (note that the function β is not a "true" admittance because it depends on the incident velocity field). Then an example is presented (section 4) for the class of problems where the boundary layers can be considered as very thin regarding the dimensions of the acoustic domain [7]. For simplicity and brevity, the example considered here is for cylindrical

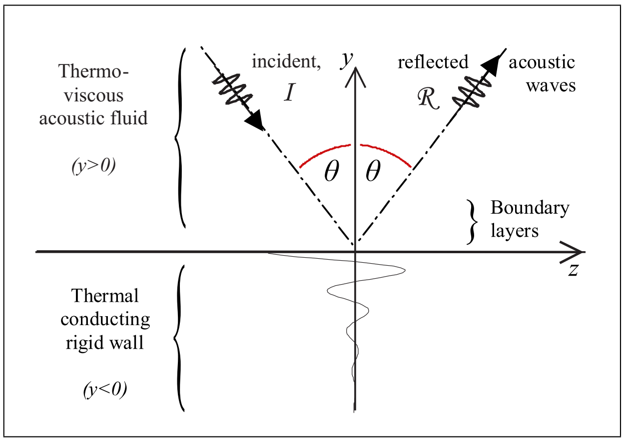

Figure 1. Incident ($\mathcal{I}$) and reflected ($\mathcal{R}$) plane-waves on a rigid thermal diffusing wall.

cavity where the cross-section is rectangular, when a plane wave field is excited by a time-periodic piston source at one end of the cavity, the other end being closed by a rigid wall.

The bounded domain considered in this section is a semi-infinite, fluid-filled domain, rigid-walled by a solid plane at rest set at $y = 0$ (Figure 1). Reactive and absorbing processes at the rigid plane boundary arise from vortical and entropic processes, assuming the non-slip condition and the continuity of both the temperature variation and the heat flow through the interface. Acoustic energy is provided in the fluid by an harmonic incident plane wave I with a direction of propagation in the $y - z$ plane inclined at an angle $\theta > 0$ to the y-axis as shown in Figure 1. This incident acoustic plane wave is represented by a field a (temperature, or density, or velocity divergence, or alternatively the acoustic pressure), in the form

$$a_i = a_0 \mathrm{e}^{-\mathrm{i}\chi_y y} \mathrm{e}^{-\mathrm{i}\alpha z} \mathrm{e}^{\mathrm{i}\omega t},$$

where, for convenience, α and χ_y are defined respectively as $\alpha = k_a \sin\theta$ and $\chi_y = k_a \cos\theta$, k_a being the complex acoustic wavenumber.

For the same variable, a reflected wave in the form

$$a_r = a_0 \mathcal{R} \mathrm{e}^{\mathrm{i}\chi_y y} \mathrm{e}^{-\mathrm{i}\alpha z} \mathrm{e}^{\mathrm{i}\omega t}$$

propagates away from the reflecting surface at an angle $\theta > 0$ to the y-axis (the angle of reflection), its complex amplitude $\mathcal{R}$ being the reflection coefficient.

The wavenumber k_a is usually given by the approximate expression

$$k_a \simeq k_0 \Big(1 - \frac{\mathrm{i}}{2} k_0 \big(\ell_v + (\gamma - 1)\ell_h \big) \Big),$$

where $k_0 = \omega / c_0$ is the adiabatic wavenumber and the factor $\ell_v + (\gamma - 1)\ell_h$ accounts for the viscous and thermal effects, except for liquids or for very high frequencies in gases, *i.e.* frequencies greater than the molecular collision frequencies (see for example [9, eq. 6.4.26] or [8, eq. 2.70]) (a more accurate expression for this acoustic wavenumber k_a is obtained in setting $\alpha = 0$ in equation (23) below for k_{ay} as a solution of equations (6)–(7) for normal plane waves).

The interaction between the acoustic wave and the rigid boundary surface is completely defined by specifying the reflection coefficient $\mathcal{R}$ as a function of the thermal- and viscous-parameters of the fluid, and the thermal parameters of the solid boundary. The reflection coefficient is related to an effective specific acoustic admittance of the boundary layer β, defined as $\beta = -\rho_0 c_0 v_{ay}/p$ where v_{ay} is the component normal to the wall of the acoustic particle velocity and p the acoustic pressure trace on the wall, by the wellknown relationship

$$\beta = \frac{\chi_y}{k_a} \frac{1 - \mathcal{R}}{1 + \mathcal{R}}. \tag{13}$$

The heat conduction contribution acts exactly the same as an additional acoustic admittance of the surface, but the viscosity contribution has a different dependence on the angle of incidence because the viscous absorption depends strongly on the component of the particle velocity parallel to the wall. The remaining of this section is devoted to express the pertinent admittance-like β which accounts for the viscous and heat diffusion effects in the fluid at the fluid-solid plane boundary $y = 0$, and for thermal diffusion in the solid.

The heat conduction equation in the solid medium ($y < 0$) can be written as follows, assuming that the diffusion process in the walls is created by the interaction between the total perturbation wave and the boundary ($y = 0$):

$$\left[(\mathrm{i}\omega\rho_w C_w + \lambda_w \alpha^2)\tau_w - \lambda_w \partial^2_{yy}\tau_w \right] \mathrm{e}^{-\mathrm{i}\alpha z} \mathrm{e}^{\mathrm{i}\omega t} = 0,$$

where ρ_w, C_w, λ_w and τ_w represent respectively the density, the heat capacity per unit mass, the isotropic thermal conductivity and the temperature variation of the wall.

The solution of this equation in the semi-infinite domain $y < 0$, which assumes that the diffusion process starts from the boundary $y = 0$, and which requires that it vanishes when y tends to (-∞), can be written as follows:

$$\tau_w(y < 0) = \tau_0 \mathrm{e}^{\sqrt{(\mathrm{i}\omega)/\xi_w + \alpha^2}\, y} \mathrm{e}^{-\mathrm{i}\alpha z} \mathrm{e}^{\mathrm{i}\omega t},$$

where $\xi_w = \lambda_w / (\rho_w C_w)$ is the thermal diffusivity of the solid material. The heat flow density through the interface $y = 0$ is then given by

$$\Phi_w = -\lambda_w \frac{\partial \tau_w}{\partial y} = -\lambda_w \sqrt{\frac{\mathrm{i}\omega}{\xi_w} + \alpha^2}\; \tau_w,$$

such that the continuity equations for the temperature variation (9) and the normal heat flux (10) reduce to the mixed boundary condition for τ at $y = 0^+$

$$R_h = \frac{\frac{\partial\tau}{\partial y}}{\tau}(y = 0^+) = \Lambda \sqrt{\frac{\mathrm{i}\omega}{\xi_w} + \alpha^2}, \tag{14}$$

where $\Lambda = \lambda_w / \lambda$ is the contrast of thermal conductivity between the rigid wall and the acoustic fluid.

Invoking equations (6-7) and considering that the coordinate x is not involved in the problem ($v_x = 0$), the particle velocity $\boldsymbol{v}$ and temperature variation τ in the fluid ($y > 0$) take the form

$$\begin{aligned} \tau &= T(y)\mathrm{e}^{-\mathrm{i}\alpha z}\mathrm{e}^{\mathrm{i}\omega t}, \\ v_y &= V_y(y)\mathrm{e}^{-\mathrm{i}\alpha z}\mathrm{e}^{\mathrm{i}\omega t}, \\ v_z &= V_z(y)\mathrm{e}^{-\mathrm{i}\alpha z}\mathrm{e}^{\mathrm{i}\omega t}, \end{aligned} \tag{15}$$

and satisfy the set of three equations which involve only the y-variable:

$$(\omega^2 + \alpha^2 A)V_y + B\partial^2_{yy}V_y - \mathrm{i}\alpha(A+B)\partial_y V_z - D\partial_y T = 0, \tag{16}$$

$$(\omega^2 - \alpha^2 B)V_z - A\partial^2_{yy}V_z - \mathrm{i}\alpha(A+B)\partial_y V_y + \mathrm{i}\alpha D T = 0, \tag{17}$$

$$(\mathrm{i}\omega - \alpha^2 Q)T + Q\partial^2_{yy}T - R\partial_y V_y + \mathrm{i}\alpha R V_z = 0, \tag{18}$$

with $A = -\mathrm{i}\omega\, c_0\, \ell'_v$, $B = \frac{c_0^2}{\gamma} + \mathrm{i}\omega c_0 \ell_v$,

$$D = \frac{\mathrm{i}\omega\hat{\beta}}{\rho_0},$$

$$Q = -\gamma \ell_h c_0, \quad R = -\frac{\gamma - 1}{\gamma\hat{\beta}}\rho_0 c_0^2.$$

Boundaries couple the actual acoustic perturbation (subscript a below) to some heat transfer (entropic movement, subscript h) and, in similar manner, to some vortical velocity (subscript v), both of which then diffuse in the fluid according to equations (16-18). It sounds therefore to sought solution as follows:

$$\begin{aligned} V_y(y) &= \mathrm{i}\alpha\left(C_{v+}\mathrm{e}^{\mathrm{i}k_{vy}y} - C_{v-}\mathrm{e}^{-\mathrm{i}k_{vy}y}\right) \\ &+ \mathrm{i}k_{hy}\left(C_{h+}\mathrm{e}^{\mathrm{i}k_{hy}y} - C_{h-}\mathrm{e}^{-\mathrm{i}k_{hy}y}\right) \\ &+ \mathrm{i}k_{ay}\left(C_{a+}\mathrm{e}^{\mathrm{i}k_{ay}y} - C_{a-}\mathrm{e}^{-\mathrm{i}k_{ay}y}\right), \end{aligned} \tag{19}$$

$$\begin{aligned} V_z(y) &= \mathrm{i}k_{vy}\left(C_{v+}\mathrm{e}^{\mathrm{i}k_{vy}y} + C_{v-}\mathrm{e}^{-\mathrm{i}k_{vy}y}\right) \\ &- \mathrm{i}\alpha\left(C_{h+}\mathrm{e}^{\mathrm{i}k_{hy}y} + C_{h-}\mathrm{e}^{-\mathrm{i}k_{hy}y}\right) \\ &+ \left(C_{a+}\mathrm{e}^{\mathrm{i}k_{ay}y} + C_{a-}\mathrm{e}^{-\mathrm{i}k_{ay}y}\right), \end{aligned} \tag{20}$$

$$\begin{aligned} T(y) &= \frac{\frac{H+G}{2Q} + \omega^2}{D}\left(C_{h+}\mathrm{e}^{\mathrm{i}k_{hy}y} + C_{h-}\mathrm{e}^{-\mathrm{i}k_{hy}y}\right) \\ &+ \frac{\frac{H-G}{2Q} + \omega^2}{D}(C_{a+}\mathrm{e}^{\mathrm{i}k_{ay}y} + C_{a-}\mathrm{e}^{-\mathrm{i}k_{ay}y}), \end{aligned} \tag{21}$$

with

$$H = DR - \omega^2 Q - \mathrm{i}\omega B, \quad G = \sqrt{H^2 - 4\mathrm{i}\omega^3 BQ}, \tag{22}$$

$$\begin{aligned} \mathrm{i}k_{vy} &= \sqrt{\frac{\omega^2}{A} + \alpha^2}, \quad \mathrm{i}k_{hy} = \sqrt{\frac{H+G}{2BQ} + \alpha^2}, \\ \mathrm{i}k_{ay} &= \sqrt{\frac{H-G}{2BQ} + \alpha^2}, \end{aligned} \tag{23}$$

and where $C_{v+}, C_{v-}, C_{h+}, C_{h-}, C_{a+}$ and C_{a-} are arbitrary constants. It must be noticed that the constants C_{v+} and C_{v-} are not involved in the solution (21) for the temperature variation τ (15), and that the divergence of the component of the velocity vector $\boldsymbol{v}$ (with the $y-$ and $z-$components given in equation 15) which is associated to constants $C_{v\pm}$ vanishes, *i.e.*

$$\mathrm{div}\left(\boldsymbol{v}(C_{h\pm} = 0, C_{a\pm} = 0)\right) = 0,$$

whereas the curl of the velocity associated to components $C_{h\pm}$ and $C_{a\pm}$ vanishes, *i.e.*

$$\mathbf{curl}\left(\boldsymbol{v}(C_{v\pm} = 0)\right) = \boldsymbol{0}.$$

This solution then highlights the vortical- ($C_{v\pm}$), entropic- ($C_{h\pm}$) and acoustic- ($C_{a\pm}$) components of the particle velocity and temperature fields, and the factors k_{vy}, k_{hy} and k_{ay} of expressions (23) are actually interpreted as vortical-, entropic- and acoustic-wavenumbers respectively, their z-component being equal for each of them to ($-\alpha$).

For the plane-wave reflection treated here, these solutions must satisfy the following conditions:

- the vortical component of the incident wave vanishes ($C_{v+} = 0$),
- the entropic component of the incident wave vanishes too ($C_{h+} = 0$),
- the amplitude of the incident acoustic wave is assumed to be equal to unity ($C_{a+} = 1$),
- both components V_y and V_z of the amplitude of the total particle velocity vanish at the fluid wall interface (non-slip Dirichlet condition), $V_y(y = 0) = V_z(y = 0) = 0$,
- the amplitude of the temperature variation satisfies the mixed condition (14), $\partial_y T(y = 0) = R_h, T(y = 0)$.

These conditions can be readily satisfied to yield the solutions of the related boundary problem. More particularly, the reflection coefficient $\mathcal{R} = C_{a-}$ and then the effective specific acoustic admittance β (equation 13) is straightforwardly obtained, yielding to

$$\begin{aligned} \beta = &\left[\mathrm{i}R_h\left(2\alpha^2 G - (H - G + 2\omega^2 Q)k_{hy}k_{vy}\right) - (H + G + 2\omega^2 Q)k_{hy}\alpha^2\right] \\ &\cdot \left[\sqrt{(H-G)/2BQ}(H + G + 2\omega^2 Q)k_{vy}R_h - \mathrm{i}\alpha^2(H - G + 2\omega^2 Q) + 2\mathrm{i}Gk_{hy}k_{vy}\right]^{-1}. \end{aligned} \tag{24}$$

This admittance-like expression (24), which vanishes when the viscous and thermal conducting effects are neglected, involves, at the fluid-solid plane interface, the effects of the viscous and thermal phenomena mentioned previously, that is mainly:

$i-$ for the acoustic fluid ($y > 0$), both the viscous effects (which depends strongly on the component of the particle velocity parallel to the wall) and the heat diffusion effects (which involve the thermal conductivity and the heat capacity of the fluid),

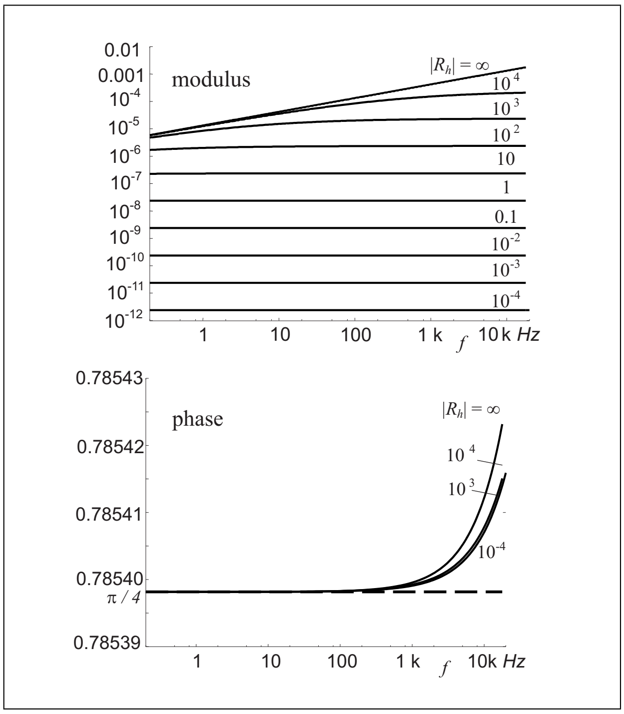

Figure 2. Admittance-like β (24) under normal incidence ($\alpha = 0$) versus frequency for air, for different values of R_h (14) ($arg(R_h) = \pi/4$).

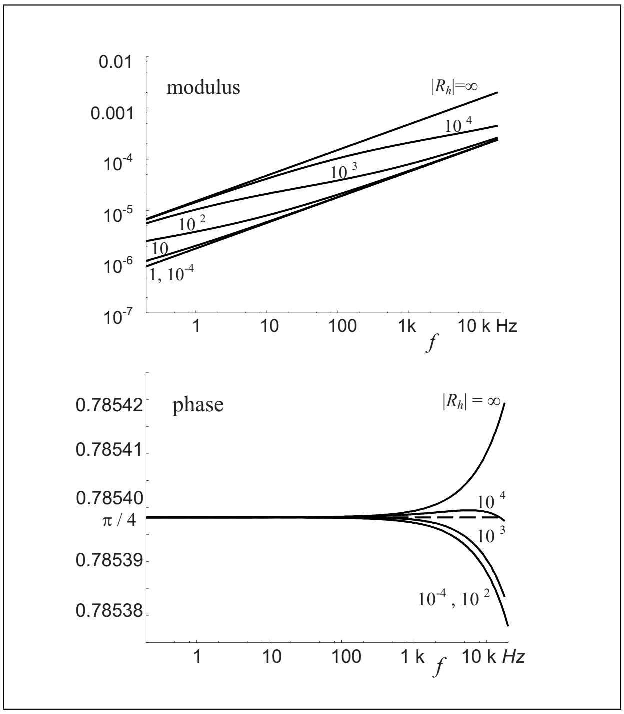

Figure 3. Admittance-like β (24) versus frequency for air, incidence 15 degrees, for different values of R_h (14) ($arg(R_h) = \pi/4$).

ii− for the wall ($y < 0$), the thermal diffusivity of the solid material of the wall.

In other words, the approach given in the present paper does not assume the isothermal condition at the wall-boundary, but accounts for the temperature variation at the wall surface due to the heat flux through the fluid-solid interface. The thermal boundary condition is usually assumed to be isothermal ($\tau \to 0, R_h \gg 1$) because of a large thermal inertia or a large thermal conductivity of the boundary relative to the thermal properties of the acoustic fluid. This is illustrated for normal incidence ($\alpha = 0$) in writing equation (14) as $R_h = \sqrt{i\omega\rho_w C_w \lambda_w}/\lambda$, showing that R_h increases with both the heat capacity and the thermal conductivity of the wall. Although the limited occurrences for which accounting for thermal diffusion in the wall is necessary, it is sufficiently broad to be of intrinsic interest and the result may be helpful when seeking precise analytic insight or numerical implementations, using the convenient dimensionless descriptor β of the wall admittance.

The contribution of the thermal diffusivity of the fluid boundary can be emphasized in presenting the modulus of the dimensionless admittance-like β as a function of the frequency f, for several values of the modulus of the factor R_h (equation 14) which highlights the role played by the thermal conductivity contrast $\Lambda = \lambda_w/\lambda$. Figure 2 presents, for air ($\rho_0 = 1.2\,\mathrm{kg\,m^{-3}}$, $c_0 = 340\,\mathrm{m/s}$, $\gamma = 1.4$, $\hat{\beta} = 371.1\,\mathrm{kg/m/s^2/K}$, $\ell_h = 6.0 \cdot 10^{-8}\,\mathrm{m}$, $\ell_v = 4.0 \cdot 10^{-8}\,\mathrm{m}$, $\ell_v' = 3.9 \cdot 10^{-8}\,\mathrm{m}$) the curves obtained for an incident acoustic wave normal to a plane boundary, the shear viscosity effect vanishing under this incidence. The upper curve ($|R_h| \to \infty$) characterizes the isothermal approximation ($\tau = \tau_w = 0$, see equation 9) at the wall boundary (this approximation is used in most problems when the fluid is a gas). On the other hand, due to the vanishing condition for the velocity on the boundary, a slight thermal conductivity contrast ($|R_h| = 0.1$) characterizes a high reflecting acoustic boundary condition ($|\beta|$ much lower than 10^{-5}) ; this behavior is increased when a deep thermal adiabatic condition occurs ($|\beta| \simeq 10^{-12}$ for $|R_h| = 10^{-4}$). Figures 3 to 5 show the same results, when the angle of incidence is respectively 15 degrees, 30 degrees and 89 degrees (grazing incidence), emphasizing the influence of the shear viscosity and showing the relative importance of the thermal contrast Λ in each situation. These results show that the expression (24) of the admittance-like β given here permits to improve the accuracy of modelling experiments. This is already the case for each mode in resonant tubes, as shown in the next section.

4. Resonant cavity with lossy boundary layers at the rigid walls

As mentioned above in the previous section, here an example is treated for the class of problems where the boundary layers can be considered as very thin regarding to the dimensions of the acoustic domain, using the expression (24) for the specific acoustic admittance β. The chosen application is a "large" (in order to allow locally near the wall the exponential solutions (19) to (21) for the vortical and entropic movements), two-dimensional rigid walled waveguide closed at one end ($z = 0$) by a plane piston

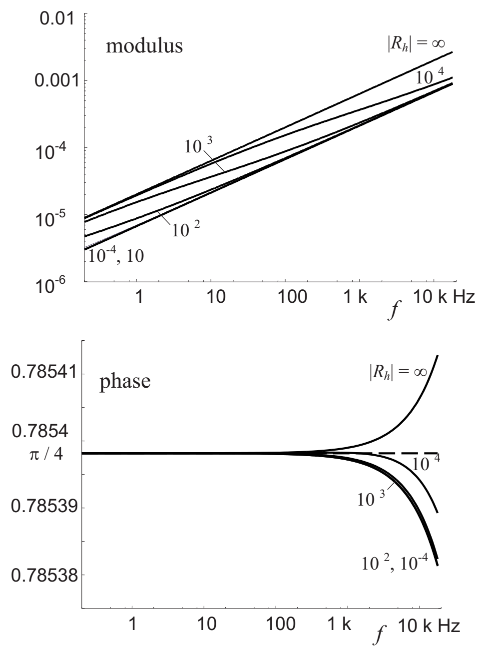

Figure 4. Admittance-like β (24) versus frequency for air, incidence 30 degrees, for different values of R_h (14) ($arg(R_h) = \pi/4$).

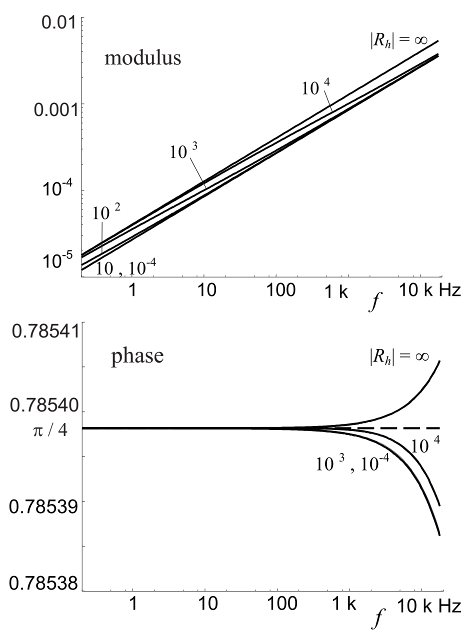

Figure 5. Admittance-like β (24) versus frequency for air, incidence 89 degrees, for different values of R_h (14) ($arg(R_h) = \pi/4$).

source (velocity $u_{z0}(x)$) and at the other end ($z = L$) by a rigid wall, the boundary lateral planes being set at the coordinates $x = 0$ and $x = \ell$ (Figure 6). The frequency range of the study lies over several resonances of the waveguide in the lower frequency range.

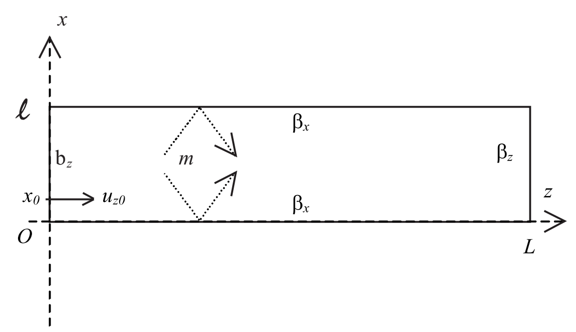

Figure 6. Resonant cavity with thermal diffusing rigid walls. Admittance-like parameters β_x and β_z are computed from expression (24), according to the incidence of each mode m and the acoustic properties of the fluid and the thermal properties of the rigid wall.

The acoustic pressure field is expressed as an eigenfunction expansion written as [8] [9]:

$$p(x,z) = \mathrm{i}\rho_0\omega \sum_{m=0}^{\infty} \left[\int_0^{\ell} u_{z0}(x_0, z_0 = 0)\psi_m(x_0)\,\mathrm{d}x_0 \right] \cdot g_m(z, z_0 = 0)\psi_m(x), \tag{25}$$

where the eigenfunctions $\psi_m(x)$ are given by

$$\psi_m(x) = \sqrt{\frac{2-\delta_{m0}}{\ell}} \cos\left(\chi_m x - \mathrm{i}\beta_x \frac{k_0}{\chi_m}\right),$$

with

$$\chi_m^2 = \left(\frac{m\pi}{\ell}\right)^2 + \mathrm{i}\left(2-\delta_{m0}\right)\frac{k_0}{\ell}2\beta_x,$$

$$\begin{cases} \delta_{m0} = 1 & m = 0, \\ \delta_{m0} = 0 & m \neq 0, \end{cases}$$

and where the Green coefficients are given by

$$g_m(z, z_0) = \frac{-\cos(k_{zm}z_0 + \gamma_z)}{k_{zm}\sin(k_{zm}L + 2\gamma_z)} \cdot \cos\left(k_{zm}(z-L) - \gamma_z\right),$$

γ_z and k_{zm}^2 being defined as $\tan\gamma_z = -\mathrm{i}(k_0/k_{zm})\beta_z$ and $k_{zm}^2 = k_a^2 - \chi_m^2$ respectively, β_x and β_z being the specific admittance β on the walls, set at $x = 0, \ell$ and at $z = 0, L$ respectively (Figure 6).

The results given in Figures 7 and 8 show respectively the amplitude (curve a) and the phase (curve b) of the acoustic pressure on the source (set at $z_0 = 0$) as functions of the frequency, when considering the two first non-plane modes ($m = 1, 2$) of the tube:

$$p_m(0,0) = -\mathrm{i}\rho_0\omega\sqrt{\frac{2}{\ell}} \cdot \frac{\cos(\gamma_z)\cos(k_{zm}L + \gamma_z)\cos(\mathrm{i}\beta_x\, k_0/\chi_m)}{k_{zm}\sin(k_{zm}L + 2\gamma_z)}. \tag{26}$$

For fluids with high thermal conductivity and low thermal-diffusing walls ($\Lambda \simeq 0.085$, table 1), these examples

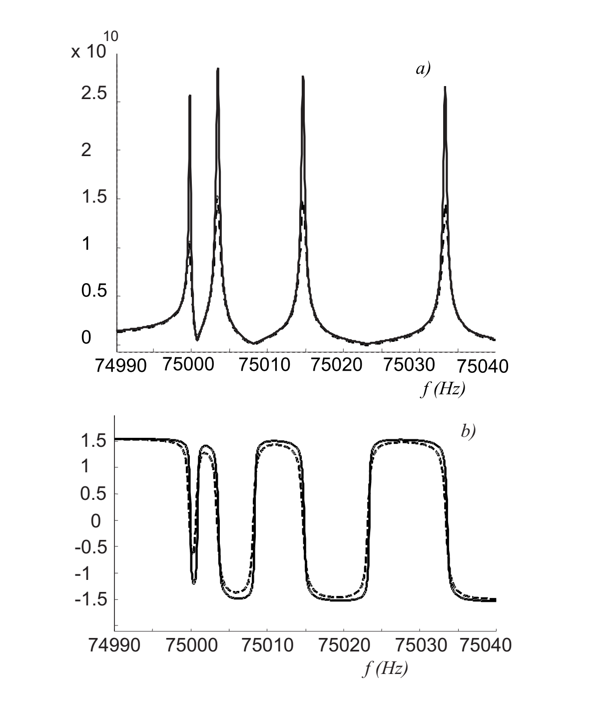

Figure 7. First resonances of the modes $m = 1$, fluid and wall properties being given in Table I, L=1 m, ℓ=0.01 m (a: amplitude, b: phase, continuous line: accounting for thermal diffusion in the wall; dotted line: isothermal approximation $|R_h| \to \infty$).

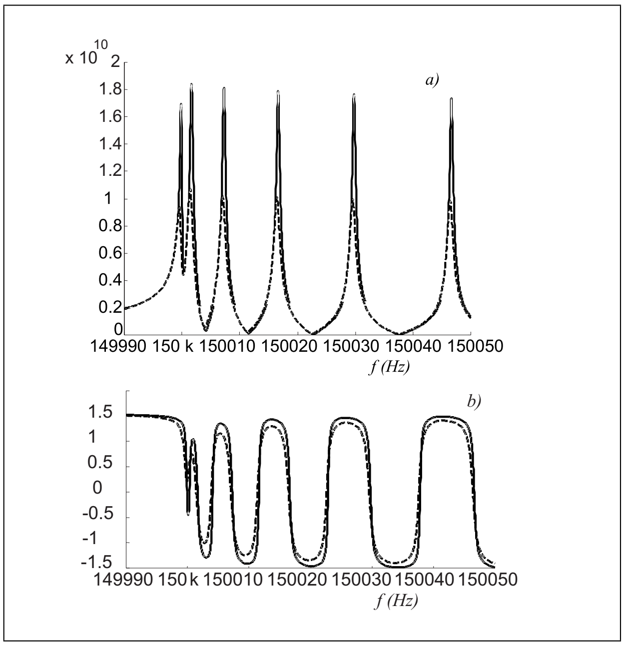

Figure 8. First resonances of the modes $m = 2$, fluid and wall properties of Table I, L=1 m, ℓ=0.01 m (a: amplitude, b: phase, continuous line: accounting for thermal diffusion in the wall; dotted line: isothermal approximation $|R_h| \to \infty$).

Table I. Properties of the rigid walls and of acoustic fluid filling the cavity of Figure 1.

	Acoustic fluid (water, 20 °C, 10^5 Pa)	Rigid wall (heat insul.)
ρ (kg m^{-}3)	998.2	0.25
Cp (J kg^{-1} K^{-1})	4174.3	2000
λ (W m^{-}1K^{-}1)	0.5984	0.05
μ (kg m^{-}1 s^{-}1)	$1.002 \cdot 10^{-3}$	
η (kg m^{-}1 s^{-}1)	neglected	
c_0 (m s^{-}1)	1500	
χT (m s^2 kg^{-}1)	$0.458 \cdot 10^{-9}$	
$\hat{\beta}$ (kg m^{-}1 s^{-}2.K^{-}1)	$4.76 \cdot 10^5$	

highlight the differences between the results obtained for the admittance-like parameter β when using (i) the equation (24) accounting for the thermal diffusion in the wall (continuous lines), and (ii) those obtained when assuming isothermal boundary conditions (dotted lines, [8] [7] and limit of equation (24) for $|R_h| \to \infty$).

The results obtained from this specific example provide confirmation of the interest of the approach presented here, in having shown the ability of the formulation (6)–(12) based on particle velocity and temperature variation to model accurately thermal boundary effects.

5. Conclusion

The examples given in the previous section show that the precise results obtained when the thermal and viscous losses play a very sensitive role provide confirmation of the relevance of the "exact" model presented in section 2 when a high precision is required. The present model (equations (6-12) allows the possibility of solving numerous problems of acoustics using standard numerical methods. Problems which can be solved include small cavities, narrow tubes and porous materials, where the thermal- and viscous- boundary layers play an important role.

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