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Fluid Layer Trapped Between a Plane, Circular Membrane and an Axisymmetrically Curved, Smooth Backing Wall: Analytical Model of the Dynamic Behaviour

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Summary
Current electrostatic transducers are designed with a planar perforated backing electrode while nowadays some authors suggest that a non planar backing wall would enhance their performances. Accurate analytic results for these new shapes of backing electrode are not available until now whereas current applications (metrology, miniaturisation) arise which need more refined modelling. Therefore the present paper aims at providing an analytical description of the strong coupling between a fluid layer trapped between an axisymmetrically curved smooth backing wall and a circularvibrating membrane (the thickness of the fluid layer depends on the radial coordinate) using modelling wherein viscosity, heat conduction, inertia and compressibility of the fluid are considered, beyond realistic boundary conditions (no slip condition and no temperature variations near the walls). Solutions are given as the sum of an expansion on eigenfunctions of the membrane *in vacuo* and an extended power series (using Frobenius method) to overcome the limitations of a standard electrical equivalent network analysis in the frequency range of interest (up to 100 kHz for circular membranes approximately one millimetre in diameter). The results obtained provide tools for optimising the design of such small electrostatic transducers.

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1. Introduction

Electrostatic transducers have been used commonly as receivers and emitters for a long time. Many theoretical and experimental results on this kind of transducers have been published in the literature (papers [1, 2, 3, 4] and textbooks [5, 6] among others) for more than five decades. More recently, numerical methods have been developped, jointly with the miniaturisation of the device on silicon chips. Authors [7, 8, 9, 10], among others, use finite element analysis to refine the calculation of the lumped element parameters by taking into account a more realistic behaviour of the mechanical subparts. However, such refinement may be limited by use of the equivalent network analysis (which implies intrinsic simplifications) and the level of discretisation for both the fluid and the structure (which requires very small elements to provide high accuracy in the acoustic boundary layers and the fluid-structure coupling, leading to an increase of the computation time).

The heart of the electrostatic transducers is a circular moving electrode (or a square electrode when fabricated using etching processes) and a backing, planar electrode with a regular array of holes, both being separated by a thin fluid layer (compared to the thermal and viscous penetration depth), the fluid being enclosed in a small cavity behind the moving electrode which is generally assumed to behave like a membrane. The stress in the membrane is carefully controlled. The geometry, the dimensions, and the position of the holes in the backing electrode are designed in such a way that the viscous damping of the system is optimal for ensuring a flat response over a wide frequency range. The thickness of the fluid layer is chosen as a compromise solution between the sensitivity of the transducer, the contribution to the damping of the diaphragm, and the disruptive voltage of the electrostatic condenser (the last one having to be avoided).

During the last decade, new designs have been proposed in which the backing electrode is non planar. In order to increase the sensitivity and even to reduce harmonic distortion originating in the cartridge, both a backing electrode shaped as a shallow dish (concave shape) [4] and a convex-shaped one [11] have been suggested. In both cases, the standard analytic procedure whereby the acoustic field (between the electrodes) and the movement of the membrane are expressed yields simple expressions for the sensitivity of the microphone, but when a precise model is needed, the limitations can be pointed out (the coupling

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between the membrane and the fluid layer being not taken into account in a very realistic way). Accordingly, when the thickness of the fluid-gap depends on the radial coordinate, it is appropriate to provide a more precise approach. It is the aim of this paper to present such an analysis that includes viscous and thermal effects in the basic propagation and diffusion equations.

The coupling between the entrapped fluid and the membrane will come both -i/ in the compressional force applied to the fluid by the vibrating membrane (changing the density of the fluid which in turn provides reaction pressure on the membrane, and then changing the temperature of the fluid so that heat exchange between the fluid and the membrane occurs) and -ii/ in the frictional force arising because the radial velocity of the entrapped fluid depends strongly on the coordinate normal to the membrane and to the backing wall (assuming non slip conditions on both).

These phenomena depend strongly on the thickness and the shape of the fluid layer. In the present paper the thickness is a function of the radial coordinate because the smooth backing wall is assumed to be continuously curved in an axisymmetric way: its profile depends on the radial coordinate in such a way that the thickness of the fluid layer increases or decreases when the radial coordinate increases, depending on the design chosen.

Therefore, the characterisation of both the fluid motion (namely the fluid flow and the pressure field inside the fluid layer), assuming an impedance boundary condition at the periphery, and the displacement field of the circular membrane (subject to the Dirichlet condition along the edge of a circular rigid frame) that occur when a time periodic, uniform pressure field is applied on the outer side of the membrane, need to construct solutions in terms of extended power series (using Frobenius method) of a problem of two coupled differential linear equations having coefficients that are analytic functions of the radial coordinate. It is the main result given in the paper.

2. The basic problem

The system under consideration comprises a thin circular membrane held under tension (T) at a very small distance away from a rigid circular backplate, both having the same radius r_c (Figure 1) (considering the moving electrode as a plate or as both a membrane and a plate would result in overly intricate formulation which would overshadow the purpose of the paper). The thickness (ε) of the compressible, heat conducting, and viscous fluid trapped between the membrane and the backplate depends on the radial coordinate r because the backplate is assumed to be continuously curved in an axisymmetric way. This thickness is much lower than both the acoustic wavelength and the radius r_c in order to ensure a quasi uniform pressure field through the fluid-gap along the z-axis (perpendicular to the membrane at rest). The displacement of the membrane vanishes along its edge ($r = r_c$) and the acoustic field in the fluid layer is subject to an impedance boundary condition at its periphery ($r = r_c$). On the membrane and the

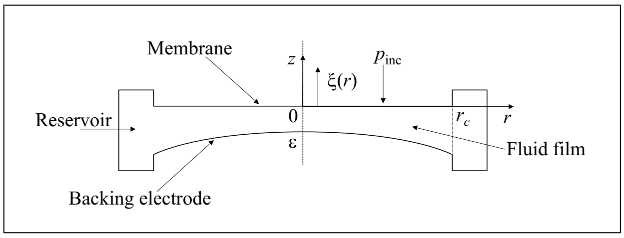

Figure 1. Schematic view of a miniaturized microphone with a non planar backing electrode and a peripheral reservoir.

backplate, realistic boundary conditions, *i.e.* no slip condition and no temperature variation, are considered, and the continuity of the velocity of the membrane and of the component of the particle velocity of the fluid normal to the membrane is accounted for. A time periodic, uniform pressure field is assumed to be applied on the outer side of the membrane.

The variables describing the dynamical and thermodynamical state of the fluid are the pressure variation p, the particle velocity $\mathbf{v}$, the density variation ρ', and the temperature variation τ. The parameters which specify the properties and the nature of the fluid are the ambient values of the density ρ, the shear viscosity coefficient μ, the bulk viscosity coefficient η, the coefficient of thermal conductivity λ_h, the heat coefficients at constant pressure and constant volume per unit of mass C_P and C_V respectively, the specific heat ratio $\gamma = C_P/C_V$, and the increase in pressure per unit increase in temperature at constant density β.

A complete set of linear homogeneous equations governing small amplitude disturbances of the fluid includes the following [12, 13]:

- the Stokes-Navier equation

$$\frac{1}{c_0}\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho c_0}\mathbf{grad}\,(p) + l_v \mathbf{grad}\,(\mathrm{div}\,(\mathbf{v})) - l'_v \mathbf{rot}\big(\mathbf{rot}(\mathbf{v})\big), \tag{1}$$

  where c_0 is the adiabatic speed of sound and l_v and l'_v are characteristic lengths defined as,

$$l_v = \frac{1}{\rho c_0}\left(\eta + \frac{4}{3}\mu\right), \quad l'_v = \frac{\mu}{\rho c_0}, \tag{2}$$

- the conservation of mass equation expressed in the domain D_0 bounded by the membrane and the backplate, taking into account the thermodynamic law expressing the density variation as a function of the independent variables p and τ,

$$\iiint_{\mathrm{D}_0} \left\{ \frac{\gamma}{c_0^2}\frac{\partial}{\partial t}(p - \beta\tau) + \rho\,\mathrm{div}\,(\mathbf{v}) \right\} \mathrm{dD}_0 = \iiint_{\mathrm{D}_0} \{\rho Q\}\,\mathrm{dD}_0, \tag{3}$$

  where Q represents the volume velocity of an external source (here the membrane),

- the Fourier equation for heat conduction, taking into account the thermodynamic law expressing the entropy variation as a function of the independent variables p and τ

$$\left[\frac{1}{c_0}\frac{\partial}{\partial t} - l_h\Delta\right]\tau = \left(\frac{\gamma-1}{\beta\gamma}\right)\frac{1}{c_0}\frac{\partial p}{\partial t} \tag{4}$$

where the characteristic length l_h is defined as

$$l_h = \frac{\lambda_h}{\rho_0 c_0 C_P}. \tag{5}$$

The acoustic pressure inside the fluid layer, which is assumed to depend only on the radial coordinate r, should be the solution of this set of three equations with the requirement of regular behaviour at the centre, this being

$$p, \mathbf{v}, \text{ and } \tau \text{ finite at } r = 0, \tag{6}$$

with an impedance boundary condition at the outer surface of the cavity ($r = r_c$), involving the r-component v_r of the particle velocity v and the acoustic pressure p,

$$p(r_c) = Z v_r(r_c), \tag{7}$$

and assuming non slip condition and no temperature variation on the membrane and the backplate, namely

$$v_r = 0, \text{ at } z = 0 \text{ and } z = \varepsilon(r), \tag{8}$$

$$\tau = 0, \text{ at } z = 0 \text{ and } z = \varepsilon(r). \tag{9}$$

Finally, the axisymmetric displacement field ξ of the circular membrane is governed by the propagation equation

$$T(\Delta_r + K^2)\xi = p_{inc} - p, \tag{10}$$

where p_{inc}, T, and K represent respectively the time periodic (angular frequency ω) pressure field applied on the outer side of the membrane, the tension of the membrane, and the wave number ($K^2 = \omega^2\mu_s/T$, μ_s being the mass per unit area of the membrane), and is subjected to the Dirichlet boundary condition at the periphery ($r = r_c$), namely

$$\xi(r = r_c) = 0. \tag{11}$$

Several hypotheses can be made here in order to avoid overly intricate formulation which would overshadow the purpose of this paper. These assumptions can be summarized as follow [12]:

- the solution does not depend on the azimuthal coordinate (the problem is assumed to be axisymmetric),
- the derivative of both the particle velocity $\mathbf{v}$ and the temperature variation τ with respect to the z-coordinate are much greater than their derivatives with respect to the radial coordinate r

$$\left|\frac{\partial \mathbf{v}}{\partial z}\right| \gg \left|\mathbf{grad_r}(\mathbf{v})\right|, \quad \left|\frac{\partial \tau}{\partial z}\right| \gg \left|\mathbf{grad}_r(\tau)\right|, \tag{12}$$

and the radial component v_r of the particle velocity is much greater than its component v_z normal to the membrane at rest

$$|v_r| \gg |v_z|, \tag{13}$$

- the derivative of the acoustic pressure with respect to the radial coordinate r is much greater than its derivative with respect to the z-coordinate

$$\left|\mathbf{grad_r}(p)\right| \gg \left|\frac{\partial p}{\partial z}\right|, \tag{14}$$

and the acoustic pressure depends only on the radial coordinate r

$$p(r, z) = p(r), \tag{15}$$

- the temperature variation τ and consequently the density variation ρ', and the z-component v_z of the particle velocity depend on both z and r coordinates because v_z and τ vanish at the walls, *i.e.* the membrane and the backplate (they are approximated below by their mean value across the section of the fluid layer (denoted $\langle ... \rangle$).

These approximations enable us to greatly simplify the expressions of equations (1) and (4), giving, for harmonic motion (angular frequency ω),

$$\left[\frac{\partial^2}{\partial z^2} + k_v^2\right] v_r = -(\rho c_0 l'_v)^{-1}\frac{\partial p}{\partial r}, \tag{16}$$

$$\left[\frac{\partial^2}{\partial z^2} + k_h^2\right] \tau = \left(\frac{\gamma-1}{\beta\gamma}\right) k_h^2 p, \tag{17}$$

where the expressions of the wave numbers k_v (associated with the vortical movement due to viscosity effects) and k_h (associated with entropy diffusion due to heat conduction) are respectively given by

$$k_v = -\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{\omega}{l'_v c_0}}, \tag{18}$$

$$k_h = -\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{\omega}{l_h c_0}}. \tag{19}$$

Therefore, it is a simple matter to solve separately these equations according to the boundary conditions (8) and (9) respectively, yielding

$$\langle v_r \rangle = -\left(\frac{1}{\mathrm{i}\omega\rho_0}\right) F_v \frac{\partial p}{\partial r}, \tag{20}$$

$$\langle \tau \rangle = \left(\frac{\gamma-1}{\beta\gamma}\right) F_h p, \tag{21}$$

with

$$F_{v,h} = \left(1 - \frac{\tan(k_{v,h}\varepsilon/2)}{k_{v,h}\varepsilon/2}\right), \tag{22}$$

the thickness ε of the fluid layer depending on the radial coordinate r. In addition, considering an elementary domain D_0 of thickness dr, bounded by the membrane and the backplate, thus enclosed in a closed surface denoted

S_0, the three terms which appear in equation (3) can be written respectively as follows:

$$\iiint_{D_0} \frac{\partial}{\partial t}(p-\beta\tau)\,\mathrm{d}D_0 \approx \frac{\partial}{\partial t}(p-\beta\langle\tau\rangle)2\pi r\varepsilon(r)\mathrm{d}r, \tag{23}$$

$$\begin{aligned}\iiint_{D_0} \rho\mathrm{div}\,(\mathbf{v})\,\mathrm{d}D_0 &= \oiint_{S_0} \rho\mathbf{v}\,\mathrm{d}\mathbf{S}_0 \\ &\approx \rho_0\frac{\partial}{\partial r}(\langle v_r\rangle 2\pi r\varepsilon(r))\,\mathrm{d}r,\end{aligned} \tag{24}$$

$$\iiint_{D_0} \rho Q\mathrm{d}D_0 = -2\pi r\,\mathrm{d}r\rho_0\frac{\partial\xi}{\partial t}, \tag{25}$$

yielding, for harmonic motion,

$$\mathrm{i}\omega\frac{\gamma}{c_0^2}(p-\beta\langle\tau\rangle)+\rho_0\frac{1}{r\varepsilon(r)}\frac{\partial(r\varepsilon(r)\langle v_r\rangle)}{\partial r} = -\mathrm{i}\omega\rho_0\frac{\xi(r)}{\varepsilon(r)}. \tag{26}$$

Finally, invoking equations (20) and (21), equation (26) leads straightforwardly to the following propagation equation inside the fluid layer:

$$\left\{\frac{\partial^2}{\partial r^2}+\left(\frac{1}{r}+\frac{1}{\varepsilon(r)}\frac{\partial\varepsilon(r)}{\partial r}+\frac{1}{F_v(r)}\frac{\partial F_v(r)}{\partial r}\right)\frac{\partial}{\partial r} +\chi^2(r)\right\}p = -\rho_0\omega^2\frac{\xi(r)}{\varepsilon(r)F_v(r)}, \tag{27}$$

where the complex wave number χ takes the form

$$\chi^2(r) = \frac{\omega^2}{c_0^2}\frac{\left(1+(\gamma-1)\dfrac{\tan(k_h\varepsilon(r)/2)}{k_h\varepsilon(r)/2}\right)}{\left(1-\dfrac{\tan(k_v\varepsilon(r)/2)}{k_v\varepsilon(r)/2}\right)}. \tag{28}$$

Due to the non uniform thickness $\varepsilon(r)$ of the fluid layer, this propagation equation has coefficients that are analytic functions of the coordinate r. The term $\partial_r F_v(r)/F_v(r)$ accounts for viscous effects which depend on the thickness of the fluid layer, the term $\partial_r\varepsilon(r)/\varepsilon(r)$ accounts for the shape of the fluid layer, and the complex expression of the wave number represent the speed of sound and the thermal and viscous dissipation effects.

3. The boundary problem and its Frobenius solution

The displacement field ξ of the membrane and the acoustic pressure p of the fluid layer (both being finite at $r=0$) are governed by the set of two coupled equations, respectively equation (10) and (27), and they are subjected to the boundary conditions (11) and (7) at the periphery ($r=r_c$).

The displacement field of the membrane

The solution for the displacement field of the membrane can be expressed as the sum of three functions as follows:

$$\xi(r) = \frac{p_{inc}-p(r_c)}{TK^2}\left(1-\frac{J_0(Kr)}{J_0(Kr_c)}\right)+\sum_{n=0}^{\infty}\xi_n\Psi_n(r), \tag{29}$$

where the first one $[p_{inc}-p(r_c)]/(TK^2)$ is the solution of the equation of the membrane when the movement is forced by the incident, homogeneous, and harmonic pressure field p_{inc} applied on the outer side of the membrane (rear the value of the acoustic pressure of the fluid layer at its periphery, $r=r_c$); the second one is the general solution $J_0(Kr)$ of the homogeneous equation of the membrane (the sum of the first two terms being subjected to the boundary conditions mentioned before), and the third one is given by an expansion on eigenfunctions ψ_n (with associated eigenvalues K_n) of the clamped membrane *in vacuo*, solution of the equation of the membrane, the right hand side being given by the pressure of the fluid layer, rear the value at its periphery ($p-p(r_c)$), leading to

$$\xi_n = -T^{-1}(K^2-K_n^2)^{-1}\langle p-p(r_c),\Psi_n\rangle, \tag{30}$$

the scalar product represented by the bracket $\langle,\rangle$ showing the coupling effect with the acoustic pressure p inside the fluid layer. Note that introducing $p(r_c)$ as presented in equation (29) and (30) permits to satisfy equation (10) for the eigenfunction expansion, third part of the solution (29), with $\Delta_r \equiv K_n$.

The acoustic pressure field inside the fluid layer

The general solution which satisfies the propagation equation (27) for the acoustic field can be written as

$$p(r) = \sum_{m=0}^{\infty} p_m\Psi_m(r)+\mathsf{P}(r), \tag{31}$$

which will be recognized as the usual solution sum of two functions, the first one being a solution of equation (27) and the second one satisfying the homogeneous equation associated to the same equation (27). Substituting this solution in the boundary condition (7), invoking equation (20) to express the radial component of the particle velocity, leads to

$$\mathsf{P}(r_c) = -(Z/\mathrm{i}\omega\rho_0)F_v(r_c)\left\{\left.\frac{\partial\mathsf{P}(r)}{\partial r}\right|_{r=r_c} +\sum_{m=0}^{\infty}p_m\left.\frac{\partial\Psi_m(r)}{\partial r}\right|_{r=r_c}\right\}. \tag{32}$$

By the use of the Frobenius method (see for example [14]), we may now give an explicit expression for the solution $\mathsf{P}(r)$ of the homogeneous equation associated to equation (27), subjected to the boundary condition (32). For compactness, in the following this homogeneous equation is written as:

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{g(r)}{r}\frac{\partial}{\partial r}+\frac{h(r)}{r^2}\right\}\mathsf{P}(r)=0, \tag{33}$$

where the coefficient $g(r)$ and $h(r)$ are expressed as follows

$$\begin{aligned}g(r) &= 1+r\frac{1}{\varepsilon(r)}\frac{\partial\varepsilon(r)}{\partial r}+r\frac{1}{F_v(r)}\frac{\partial F_v(r)}{\partial r} \\ &= \sum_{k=0}^{\infty}g_k r^k,\end{aligned} \tag{34}$$

$$h(r) = \chi^2(r) r^2 = \sum_{k=0}^{\infty} h_k r^k, \quad (35)$$

the power series being known when the shape of the backing electrode is given (k is an integer).

Therefore it is a simple matter to solve this equation, expressing $\mathsf{P}(r)$ as a power series expansion:

$$\mathsf{P}(r) = \sum_{k=0}^{\infty} \mathsf{P}_k r^{k+q} = r^q \sum_{k=0}^{\infty} \mathsf{P}_k r^k. \quad (36)$$

Substituting this expression in equation (33) lead straightforwardly to the indicial and recurrence equations:

$$q^2 + (g_0 - 1)q + h_0 = 0, \quad (37)$$

$$\mathsf{P}_m(m+q)(m+q-1) + \sum_{k+j=m} ((k+q)g_j + h_j)\mathsf{P}_k = 0. \quad (38)$$

For the expressions of $g(r)$ and $h(r)$ considered here (next section), only one root of the first equation remains finite when the coordinate r vanishes (it is denoted q_1 below). Thus equations (37) and (38) lead to (for $k = 1, 2, 3, \ldots$)

$$\mathsf{P}_k = -\frac{1}{k(2q_1 + k + g_0 - 1)} \cdot \left(\sum_{l=0}^{k-1} \big((l + q_1)g_{k-l} + h_{k-l}\big)\mathsf{P}_l \right). \quad (39)$$

Then, substituting this result in the solution (36) and taking into account the boundary condition (32) lead to

$$\mathsf{P}(r) = \sum_{m=0}^{\infty} p_m \gamma_m(r), \quad (40)$$

with

$$\gamma_m(r) = -\frac{(Z/\mathrm{i}\omega\rho_0)F_v(r_c)}{r_c^{q_1} \sum_{k=0}^{\infty} \left(r_c^k + (Z/\mathrm{i}\omega\rho_0)F_v(r_c)(k+q_1)r_c^{k-1}\right) \mathsf{P}_k} \cdot \left.\frac{\partial \Psi_m(r)}{\partial r}\right|_{r=r_c} \sum_{k=0}^{\infty} \mathsf{P}_k r^{k+q_1}, \quad (41)$$

this expression being known when the set of equation (39) is solved.

Finally, the solution for the acoustic pressure p in the fluid layer takes the form

$$p(r) = \sum_{m=0}^{\infty} p_m \left[\Psi_m(r) + \gamma_m(r)\right], \quad (42)$$

the unknown coefficients p_m being obtained from the propagation equation (27), substituting the expressions (29) and (30) to the displacement ξ of the membrane in the right hand side and making use of the orthogonality property of the eigenfunctions ψ_n.

A lengthy but straightforward calculation leads to the set of linear algebraic equations giving the expression of the coefficients p_m and then the expression of coefficients ξ_n (equation (30)), namely, $\forall k \in \mathbb{N}$,

$$\sum_{m=0}^{\infty} \left(\phi_{km} + \sum_{n=0}^{\infty} \mathsf{C}_{kn} T^{-1}(K^2 - K_n^2)^{-1}(\delta_{nm} + \Gamma_{nm}) \right) p_m = S_k, \quad (43)$$

$$\xi_n = -T^{-1}(K^2 - K_n^2)^{-1} \sum_{m=0}^{\infty} (\delta_{nm} + \Gamma_{nm}) p_m, \quad (44)$$

where

δ_{nm} is the delta function, (45)

$$\phi_{km} = \int_{\mathsf{S}} \left\{ \left(\frac{g(r) - 1}{r} \right) \frac{\partial \Psi_m(r)}{\partial r} + \left(\chi(r)^2 - K_m^2\right)\Psi_m(r) + \frac{\gamma_m(r_c)}{TK^2} \left(1 - \frac{J_0(Kr)}{J_0(Kr_c)} \right) \right\} \Psi_k(r) \mathrm{d}\mathsf{S}, \quad (46)$$

where S is the surface of the membrane,

$$\Gamma_{nm} = -\frac{(Z/\mathrm{i}\omega\rho_0)F_v(r_c)}{\mathsf{P}(r_c) + (Z/\mathrm{i}\omega\rho_0)F_v(r_c) \left.\frac{\partial \mathsf{P}(r)}{\partial r}\right|_{r=r_c}} \cdot \left.\frac{\partial \Psi_m(r)}{\partial r}\right|_{r=r_c} \left\langle \mathsf{P} - \mathsf{P}(r_c), \Psi_n \right\rangle, \quad (47)$$

$$\mathsf{C}_{kn} = \left\langle \zeta \Psi_n, \Psi_k \right\rangle, \quad (48)$$

$$S_k = \frac{p_{inc}}{TK^2} \left\langle \zeta \left(1 - \frac{J_0(Kr)}{J_0(Kr_c)} \right), \psi_k \right\rangle, \quad (49)$$

with

$$\zeta = -\rho\omega^2/(\varepsilon F_v), \quad (50)$$

the summation over the integer m being restricted, in practice, to the lower order modes, depending on the dimensions of the fluid layer and the wavelengths considered.

These results are consistent with those which have been obtained from a previous model presented for a fluid layer of constant thickness [15]. For example, if the impedance of the outer surface of the fluid layer vanishes (leading to $p(r_c) = 0$), the contribution of the n^{th} mode to the displacement field of the membrane, when ε does not depend on the radial coordinate r, reduces to

$$\left\langle \xi, \Psi_n \right\rangle \Psi_n = p_{inc} \frac{2}{K_n r_c} \frac{1}{T(K^2 - K_n^2) + \zeta(\chi^2 - K_n^2)^{-1}} \cdot \frac{J_0(K_n r)}{J_1(K_n r_c)}. \quad (51)$$

4. Results and discussion

In the remainder of the paper, the results can be given for any shape of the backing electrode, the thickness of the fluid layer being described in terms of a power series of the radial coordinate, but to facilitate physical insight into

Table I. Values of the main parameters used.

Fluid (air)	thermal charact. length viscous charact. length specific heat ratio density adiab. speed of sound	l_h l'_v γ ρ_0 c_0	$5.6 \cdot 10^{-8}$ m $4.5 \cdot 10^{-8}$ m 1.4 1.2 kg/m^3 340 m/s
Geometry	radius of the membrane volume of the peripheral cavities	r_c V_{cav}	1.5 mm $5 \cdot 10^{-10}$ m^3
Membrane (silicon)	tension density thickness	T ρ_{Si} h_m	600 N/m 2330 kg m^{-3} 10 μm

dynamic behaviour of such system, being given the scope of this paper, the shape selected is parabolic which yields specific results, yet representative of the general conclusions. The thickness of the fluid layer ε is then written as

$$\varepsilon(r) = \varepsilon_0 \left(1 \pm \alpha^2 (r/r_c)^2\right), \qquad (52)$$

where ε_0 is the thickness at $r = 0$ and α is a parameter which governs the thickness variation of the fluid layer from the centre ($r = 0$) to the entrance of the peripheral cavity ($r = r_c$), the symbol $\pm$ accounting for a convex or a concave shape for the backing electrode.

First, it is worth noting that accurate approximate expressions can be substituted to expression (22) of the functions $F_{v,h}$, both in the lower frequency range (in the limit of small values of $|k_{v,h}\varepsilon|$) and in the higher frequency range (asymptotic behaviour for large values of $|k_{v,h}\varepsilon|$), being given the usual thickness ε of such components (here $15 \mu m < \varepsilon < 30\,\mu$m). In the intermediate frequency range, namely here roughly from 10 kHz to 1 MHz, the expression (22) of $F_{v,h}$ must be used without approximation, because in this frequency range, expressions (18) and (19) of the complex wavenumbers k_v and k_h respectively, which appear in the expression of the functions F_v and F_h, lead to $1 < |k_{v,h}\varepsilon| < 10$ for $\varepsilon = 18\,\mu$m and for the values of the parameters given in table I.

In addition, we would like to emphasize that the terms which account for the shape of the fluid layer in equation (27), namely $(r\partial_r F_v/F_v)$ and $(r\partial_r \varepsilon/\varepsilon)$ in the function g (equation (34)), are not much smaller than 1.

In order to evaluate the influence of these corrections due to both viscous effects and the shape of the fluid layer, their absolute values are shown in Figure 2, as function of the radial coordinate r, for given frequencies and for a given parabolic convex shape, namely $\varepsilon_0 = 18\,\mu$m and $\alpha = 0.6$ (which induces $\varepsilon(r_c) = 29.5\,\mu$m at the periphery). These parameters vanish at $r = 0$ but, in the frequency range of interest, they become quickly of the same order of magnitude as 1 when r increases. These results show that the wave propagation operator is deeply constrained by the variation of thickness along with the r-coordinate, through the shape itself but also through the variation of the viscous contribution. Likewise, the continuous variations in geometric, viscous, and thermal properties must

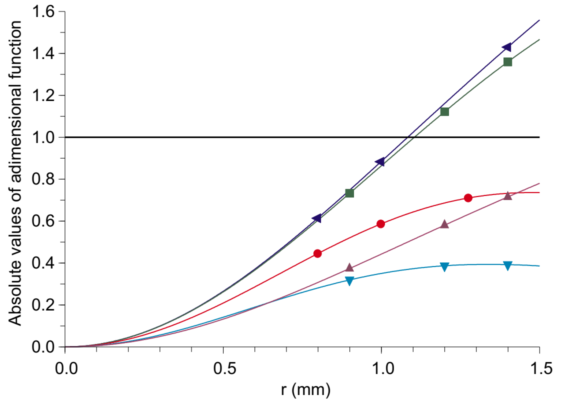

Figure 2. Absolute values of the parameter $(r\partial_r F_v/F_v)$ for given frequencies (1 kHz (◀), 10 kHz (■), 50 kHz (▲), 100 kHz (▼)) and the parameter $(r\partial_r \varepsilon/\varepsilon)$ (●), as functions of the radial coordinate r (mm), for a parabolic shape of the backing electrode ($\varepsilon_0 = 18\,\mu$m, $\alpha = 0.6$). These curves must be compared to 1 (horizontal straightline).

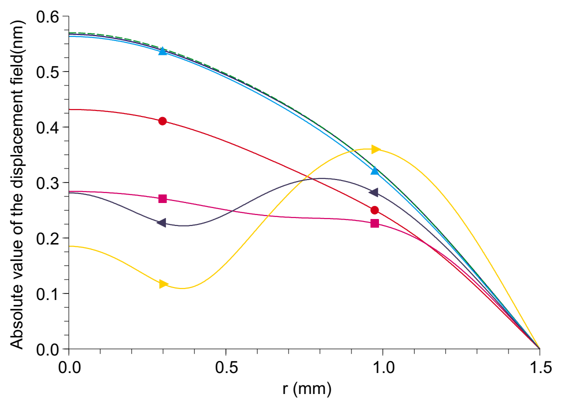

Figure 3. Absolute value of the displacement field for a circular membrane coupled with a thin fluid layer, for a planar electrode ($\varepsilon_0 = 18\,\mu$m, $\alpha = 0$), as a function of the radial coordinate r, for a set of given frequencies: 1 kHz (—), 10 kHz (- -), 20 kHz (▲), 40 kHz (●), 60 kHz (■), 80 kHz (◀), 100 kHz (▶), and for an incident pressure $p_{inc} = 1$ Pa.

be accounted for in the complex wavenumber χ and in the function εF_v in the right hand side of the propagation equation.

The absolute value of the displacement field of the membrane, shown in Figures 3 and 4 for an incident pressure $p_{inc} = 1 Pa$, respectively for a planar backing electrode ($\varepsilon_0 = 18\,\mu$m, $\alpha = 0$) and a convex one ($\varepsilon_0 = 18\,\mu$m, $\alpha = 0.4$), is a quantity of interest (as mentioned below in the conclusion [16, 17]).

As expected, in the lower frequency range, up to the first resonance frequency of the membrane coupled with the fluid layer (approximately 40 kHz), the displacement of the membrane is clearly driven by the first mode of the membrane loaded by the fluid layer. But it is no more the case beyond the first resonance frequency; therefore, it may be pointed out, when using this transducer as an emitter, that the shape of the displacement field would drastically decrease its performances.

In order to reveal the behaviour of such electrostatic transducer when it is used as a receiver, the sensitivity, defined as the ratio of the output open circuit voltage and the incident acoustic pressure, is analysed as a function of the frequency. proportional to the value of the ratio (ξ/ε) which accounts for the variation of the capacitance. Thus, according to this property for a transducer with a non planar backing electrode, the sensitivity takes the form

For a transducer with a non planar backing electrode, the sensitivity takes the form

$$\sigma = \frac{U_0}{p_{inc}} \frac{\int_S \left(\frac{\xi}{\varepsilon^2}\right) \mathrm{dS}}{\int_S \left(\frac{1}{\varepsilon}\right) \mathrm{dS}}, \tag{53}$$

U_0 being the polarisation voltage and S the surface of the membrane. Equations (29), (42), (43), (44), (49) show that the displacement field ξ is proportional to the incident pressure p_{inc} which enables us to define an adimensional sensitivity σ_{ad} as

$$\sigma_{ad} = \frac{p_{ref}}{U_0}\sigma = \frac{p_{ref}}{p_{inc}} \frac{\int_S \left(\frac{\xi}{\varepsilon^2}\right) \mathrm{dS}}{\int_S \left(\frac{1}{\varepsilon}\right) \mathrm{dS}}, \tag{54}$$

where p_{ref} is a reference pressure (1 Pa). The adimensional sensitivity σ_{ad} is shown in Figure 5 as a function of the frequency, for a planar backing electrode ($\varepsilon_0 = 18\,\mu$m and $\alpha = 0$), the modal expansion being truncated at the third mode (the contribution of the upper modes appears here negligible). In these results, the input impedance $Z = p(r_c)/v_r(r_c)$ of the reservoir has been expressed as follows [12]

$$Z = \frac{\gamma P_0 S_c}{\mathrm{i}\omega V_c}, \tag{55}$$

where V_c and S_c are respectively the volume of the reservoir and its area at its entrance, and P_0 the static pressure (assumed to be here 10^5 Pa). The set of curves shows the influence of the viscous and thermal effects separately, assuming that either the viscous or the thermal penetration depth is small compared to the thickness of the fluid layer. In the lower frequency range, thermal conduction has a small effect which increases the sensitivity (because the pressure variation rear the membrane decreases when the movement is no more adiabatic).

In the vicinity of the first two resonances, the damping is mainly driven by the viscous effects. In the first valley, thermal conduction has a significant effect. In the higher frequency range (beyond 50 kHz) thermal damping reaches the same order of magnitude as viscous damping. Thus, thermal effects, which simply consist of an additional factor in the polytropic compressibility coefficient, should not be neglected to accurately describe the behaviour of such transducer for frequencies greater than roughly 10 kHz.

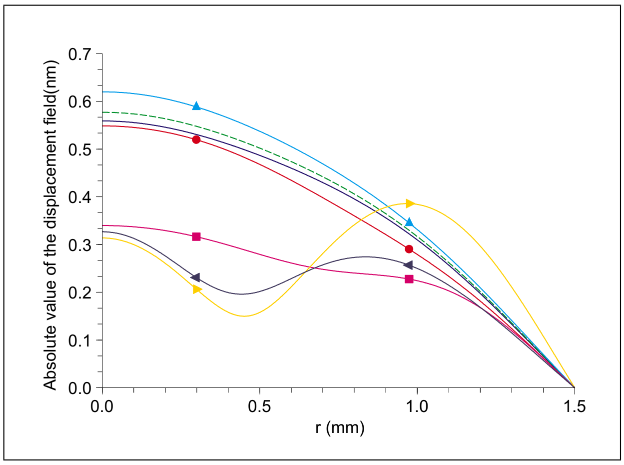

Figure 4. Absolute value of the displacement field for a circular membrane coupled with a thin fluid layer, for a parabolic backing electrode ($\varepsilon_0 = 18\,\mu$m, $\alpha = 0.4$), as a function of the radial coordinate r for set of given frequencies, 1 kHz (—), 10 kHz(- -), 20 kHz (▲), 40 kHz (●), 60 kHz (■), 80 kHz (◀), 100 kHz (▶), and for an incident pressure $p_{inc} = 1$ Pa.

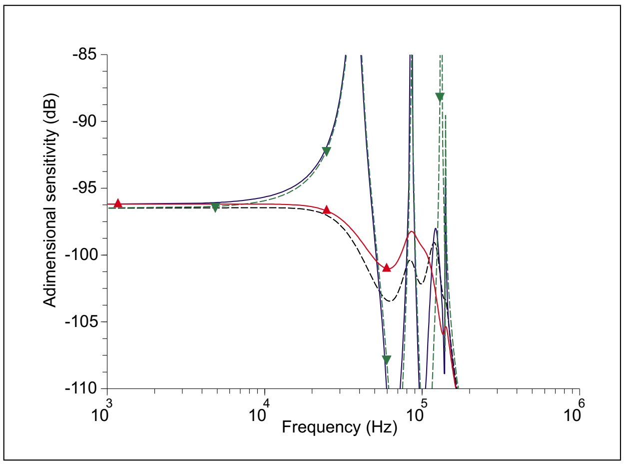

Figure 5. Influence of the viscous and thermal effects on the adimensional sensitivity of the transducer, as a function of the frequency, for a planar backing electrode ($\varepsilon_0 = 18\,\mu$m, $\alpha = 0$): without thermal effects and without viscous effects (-▼-), with thermal effects and with viscous effects (▲), with thermal effects and without viscous effects (—), without thermal effects and with viscous effects (- -).

The variation of the sensitivity as a function of the frequency for several values of the shape factor α, running from 0 to 0.8 for both a convex and a concave shape of the backing electrode, is shown in Figure 6. A key frequency (around 50 kHz) can be pointed out which appears to be the upper limit of the frequency bandwidth.

In the lower frequency range, the adimensional sensitivity for transducers with a convex (resp. concave) backing electrode (for the same values of the parameter α as above) decreases (resp. increases); this is mainly due to the decreasing (resp. increasing) of the acoustic impedance at the entrance of the peripheral reservoir, which depends on the area of this entrance at $r = r_c$ (55). Moreover, as already mentioned, both viscous and thermal effects, which

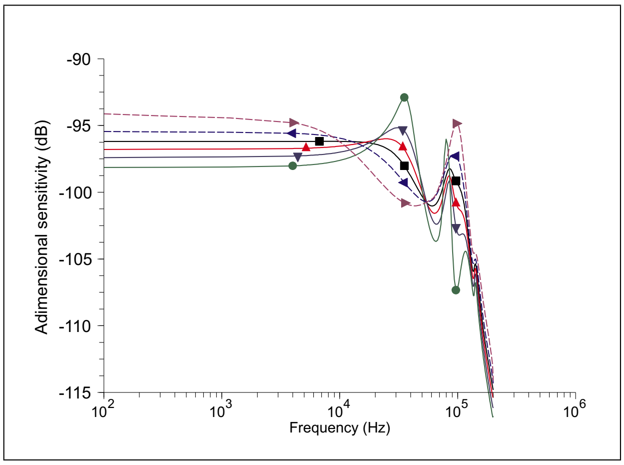

Figure 6. Adimensional sensitivity as a function of the frequency, for several parabolic shapes of the backing electrode with ε_0 = 18 μm and α running from 0 to 0.8, and for both a convex (subscript n) and a concave shape (subscript u), each configuration being denoted by S_α^n or S_α^u: $S_{0.4}^u$ (▶), $S_{0.2}^u$ (◀), S_0^n (□), $S_{0.4}^n$ (▲), $S_{0.6}^n$ (▼), $S_{0.8}^n$ (●).

depend strongly on the variation of the fluid layer thickness, drive drastically the adimensional sensitivity of the transducer for frequencies greater than 10 kHz.

General trends which would help the optimisation of the shape of the backing electrode may be found by use of the numerical tool developed to satisfy given specifications.

5. Conclusion

The original motivation for this study was to characterise the displacement field of a circular membrane when a reaction pressure on this membrane is provided by the motion of a fluid layer with tapered thickness (subjected to an impedance boundary condition at its periphery). A general approach, based on the exact description of the strong coupling between the membrane and the fluid layer, is used for the analysis of the motion of both the non-uniform fluid layer and the membrane (it includes viscous and thermal effects which depend strongly on the thickness profile of the fluid layer, among others). The solution of the problem of two coupled differential linear equations having coefficients that are analytic functions of the radial coordinate is derived using modal expansion and extended power series (Frobenius method).

The main contribution of the paper is thus a systematic framework to estimate accurately both the frequency dependent sensitivity of this kind of electrostatic transducers and the shape of the displacement field of the membrane for a given external pressure field, in the complex coupling mentioned above. The system considered here is described with a refinement that is consistent with the requirements of the devices which could appear in a near future, when miniaturisation (using MEMS technique) and very large bandwidth (up to 100 kHz, even more) would be required. Moreover, this theoretical refinement would be appropriate when precise analysis of both the shape and the local, r-dependant impedance is needed (as for example in the reciprocity calibration technique [16]) or in the measurement of thermo-physical properties in a gas-filled spherical resonator wherein acoustic resonances occur [17], among others. Actually, in the lower frequency range (up to 20 kHz for example) and for usual dimensions (half or quarter inches), lumped circuit elements that could be derived assuming appropriate approximations would provide accurate modelling in the usual situations (such modelling is beyond the scope of this paper).

The theoretical results presented in this paper convey an interpretation of physical phenomena, giving the role played by several parameters as mentioned above. These results for the sensitivity of the microphone and for the shape profile of the membrane, assuming more particularly a fluid layer with parabolic tapered thickness, are obtained in a straightforward manner, irrespective of frequency. Thus finally, requirements that have to be taken into account in the design of miniaturised (or not) electrostatic microphones (or other kind of transducers) can now be addressed with a very good accuracy, using the theoretical results obtained in this work.

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