Acta Acustica united with Acustica
Vol. 101 (2015) 859 – 862

DOI 10.3813/AAA.918880

Acoustic Fields in Thin Fluid Layers Between Vibrating Walls and Rigid Boundaries: Integral Method

Petr Honzík[1,2)], Michel Bruneau[1)]
[1)] Laboratoire d'Acoustique de l'Université du Maine, UMR CNRS 6613, Université du Maine, Avenue Olivier Messiaen, 72 085 Le Mans, France.
[2)] Czech Technical University in Prague, Faculty of Transportation Sciences, Konviktská 20, 110 00 Praha 1, Czech Republic. Petr.Honzik@gmail.com

PACS no. 43.20.Bi, 43.20.Tb, 43.38.Bs

Abstract

An exact representation, which involves integral formulation, is established for the pressure field inside a thin layer of thermo-viscous fluid of finite extent, trapped between a vibrating wall (two-dimensional membrane or plate, herein square) and a planar boundary. Thermo-viscous boundary layer effects and coupling effects are taken into account. The representation would enable analytical estimates to be found for the displacement field of the vibrating wall and pressure variations, then the sensitivity when considering electroacoustic receiver (MEMS microphones, in a large frequency range), among other applications, and would provide results hitherto resolved only by numerical or multi-modal analysis.

1. Introduction

The formulation presented in this paper would be appropriate to addressing specific devices, when the heart of the device comprises a two-dimensional thin structure of finite extent. The structure is a thin layer of thermo-viscous fluid of finite extent, trapped between a vibrating wall and generally a discrete non-planar boundary, herein a planar boundary for the sake of simplicity. The peripheral boundary is described by its impedance. The fluid-gap loads the vibrating wall which in turn acts as a non-uniformly distributed time-periodic source. The vibrating wall is a two-dimensional thin elastic plate or a membrane, herein a square membrane for the sake of simplicity, excited by an external source (incident field or others). This structure permits to handle various situations, including damping of panels, miniaturised transducers and so on.

In the literature, beyond finite element method for characterising such devices [1, 2, 3, 4], the construction of analytical solutions for such problems is made in terms of eigenmodes of both the vibrating wall and the fluid-filled elements loading this wall such as fluid-gap, backing or peripheral cavities, along with in terms of general solutions of the associated homogeneous propagation equations (see for example [4, 5] and references contained therein, [6] which do not assume uniform pressure field across the fluid-gap, [7] which give a large survey of the state of the art including discussions on hypotheses and results for several geometries). These solutions make respectively use of

Received 23 January 2015,
accepted 26 March 2015.

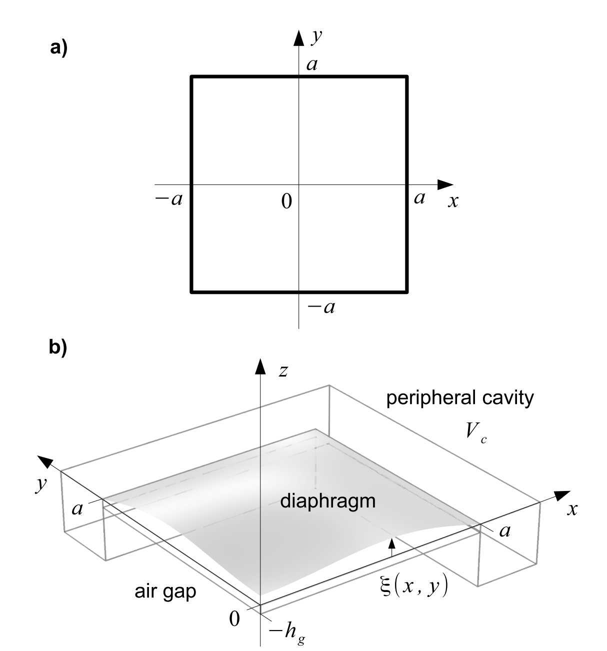

Figure 1. Geometry of the system: a) the dimensions of the square diaphragm and b) geometry of the transducer in the 1st quadrant.

modal functions which usually satisfy Dirichlet boundary conditions on the periphery of a membrane and modal functions which satisfy interface conditions or Neumann boundary conditions on the periphery of the fluid-filled elements, beyond other realistic boundary conditions such as no slip conditions and no temperature variations near the walls, the modal sums being considerably simplified (not completely summed). This conventional representation exhibits procedural difficulties due to the coupling of Dirichlet and Neumann eigenfunctions which restrict merely its use in commonly encountered situations, thereby motivating the present attempt to achieve an alternate procedure.

The alternate procedure presented herein to express analytically the solutions for the fluid-filled elements such as fluid-gap, backing or peripheral cavities is to seek to make use of Green's functions depending on two coordinates and to not express such Green's functions as sums over eigenfunctions. Thus, an exact representation which involves integral formulation is established for the pressure field inside the fluid-filled element generated by individual modes of displacement field on the vibrating wall, thermo-viscous boundary layer effects and coupling effects being actually taken into account. The representation enables analytical estimates to be found for both the displacement field of the wall and the pressure variations in the fluid layer, then for example for the sensitivity when considering electroacoustic receivers (in a large frequency range, typically more than 100 kHz for MEMS microphones) or of the damping panel, and would provide results hitherto resolved only by numerical or multi-modal analysis.

2. Analytical solutions for the coupled system

The problem addressed in this section deals with the square geometry of the structure, the external dimensions of the membrane are fixed at the coordinates $x = (-a, a)$ and $y = (-a, a)$ (Figure 1). The thickness of the fluid film trapped between the membrane and the backing electrode, denoted h_g, has the same order of magnitude as the thickness of the viscous and thermal boundary layers, the pressure variation being assumed to be constant through the thickness of the fluid film (it does not depend on the z coordinate). The peripheral cavity is defined by its volume V_c. It is worth noting that the analytical approach presented below can be readily modified to yield the analytical formulation and solutions for the other structures mentioned in the introduction.

2.1. Equations governing the acoustic pressure inside the air gap

Assuming that the diaphragm behaves as an extended source described by the z-component of its volume velocity per unit volume $U(x,y)$ positive when directed along the z-axis, the propagation in each fluid-filled element is given by

$$(\partial^2_{xx} + \partial^2_{yy} + \chi^2)p(x,y) = -U(x,y), \qquad (1)$$

where the source term $U(x,y) = \rho_0\omega^2\xi(x,y)/(h_g F_v)$ (U vanishing outside the membrane) depends on the displacement of the diaphragm, $\xi(x,y)$, and where the complex wavenumber χ accounts for the angular frequency ω of the field and the properties of the fluid, namely the compressibility and the density ρ_0 through the adiabatic speed of sound c_0, the heat capacity at constant pressure per unit of mass C_p, the specific heat ratio γ, the shear viscosity coefficient μ, and the thermal conduction coefficient λ_h (see for example [5]): $\chi^2 = (\omega^2/c_0^2)[1+(\gamma-1)(1-F_h)]/F_v$, with $F_{h,v} = 1-\tan(k_{h,v}h_g/2)/(k_{h,v}h_g/2)$, where $k_v = [(1-j)/\sqrt{2}]\sqrt{\rho_0\omega/\mu}$ and $k_h = [(1-\mathrm{j})/\sqrt{2}]\sqrt{\rho_0\omega C_P/\lambda_h}$ are the wavenumbers respectively associated with the vortical movement due to viscosity effects and with the entropic movement due to conduction effects (with a time factor given by $e^{j\omega t}$).

The acoustic pressure inside the peripheral cavity $p_c = 4w_i Z_c$, Z_c being the impedance of the peripheral cavity and w_i the volume velocity on the ith side of the periphery of the square air gap (assuming that all the volume velocities w_i has the same value), is supposed to be uniform,

$$p_c = p(a,y) = p(-a,y) = p(x,a) = p(x,-a). \qquad (2)$$

Then, the usual relation between the particle velocity and the spatial derivative of the acoustic pressure in the air gap is used to obtain the normal derivative of the pressure at the periphery of the air gap

$$\partial_n p = -\frac{\mathrm{j}\omega\rho_0}{8F_v Z_c h_g a} p_c. \qquad (3)$$

2.2. Integral formulation for the acoustic pressure

The Green's function associated with the propagation equation (1) takes the form

$$g(x,x_0;y,y_0) = -\mathrm{j}H_0^-\left(\chi|\vec{r}-\vec{r}_0|\right)/4, \qquad (4)$$

with $|\vec{r}-\vec{r}_0| = \sqrt{(x-x_0)^2+(y-y_0)^2}$, $H_n^-(z)$ denoting the Hankel function of the second kind of order n. For the reason of symmetry the acoustic pressure can be searched for only in the 1st quadrant (for $x>0$, $y>0$, see Figure 1). In this case the Green's function satisfying the Neumann's boundary condition (the first normal derivatives vanish) at $x=0$ and $y=0$,

$$\begin{aligned} G(x,x_0;y,y_0) \;=\; & g(x,x_0;y,y_0) + g(x,-x_0;y,y_0) \\ & + g(x,x_0;y,-y_0) + g(x,-x_0;y,-y_0), \end{aligned} \qquad (5)$$

has to be used. The integral formulation inside the domain, taking into account Equations (2) and (3), can be then written as

$$p(x,y) = \iint\limits_{(0,a)\times(0,a)} G(x,x_0;y,y_0)U(x_0,y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0 - p_c I_G(x,y), \qquad (6)$$

with

$$\begin{aligned} I_G(x,y) \;=\; & \frac{\mathrm{j}\omega\rho_0}{8F_v Z_c h_g a}\left[\int_0^a G(x,x_0;y,a)\mathrm{d}x_0 \right. \\ & \left. + \int_0^a G(x,a;y,y_0)\mathrm{d}y_0\right] \\ & + \int_0^a \partial_{y_0} G(x,x_0;y,a)\mathrm{d}x_0 \\ & + \int_0^a \partial_{x_0} G(x,a;y,y_0)\,\mathrm{d}y_0. \end{aligned} \qquad (7)$$

The acoustic pressure inside the peripheral cavity can be approximately calculated as the mean value of the acoustic pressure at one of the air gap periphery $p_c = \langle p(x,a)\rangle_x$ (here for example at $y=a$; the symbol $\langle\cdot\rangle_x$ denoting the mean value over the x-coordinate) which, after using Equation (6), leads to

$$\begin{aligned} p_c \;=\; & \left[1/\left(1+\left\langle I_G(x,a)\right\rangle_x\right)\right] \\ & \cdot \iint\limits_{(0,a)\times(0,a)} \left\langle G(x,x_0;a,y_0)\right\rangle_x U(x_0,y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0. \end{aligned} \qquad (8)$$

Alternatively, the value of the acoustic pressure at the corner of the air gap $p_c \approx p(a,a)$ can be used in the lower frequency range in order to reduce the number of integrals handled numerically.

2.3. Coupling between the diaphragm displacement field and the acoustic pressure field inside the air gap

The solution for a square membrane is presented in this section as an example, the solution for a plate being obtained analogously. The forced vibrations of the membrane driven by an incident acoustic pressure p_{inc}, which is the time-periodic harmonic source uniform over the membrane surface, and loaded by the acoustic pressure in the air gap $p(x,y)$, are expressed by an expansion on a classical (Dirichlet) set of orthogonal functions with associated eigenvalues k_{mn} as [5]

$$\xi(x,y) = \sum_m\sum_n \xi_{mn}\psi_{mn}(x,y), \qquad (9)$$

$$\begin{aligned} \xi_{mn} \;=\; & \frac{1}{T(k_{mn}^2 - K_M^2)} \\ & \cdot \iint\limits_{(-a,a)\times(-a,a)} \psi_{mn}(x,y)\left[p(x,y)-p_{inc}\right]\mathrm{d}x\,\mathrm{d}y, \end{aligned} \qquad (10)$$

where $K_M^2 = \omega^2 M_S/T$, T being the tension of the membrane and $M_S = h_d\rho_d$ the mass per unit area (h_d and ρ_d are the thickness of the diaphragm and its density, respectively).

Introducing the acoustic pressure in the air gap (Equation 6) into Equation (10) gives

$$T(k_{mn}^2 - K_M^2)\xi_{mn} = c_{mn} + \sum_q\sum_r \xi_{qr}A_{(mn),(qr)}, \qquad (11)$$

or, in the matrix form,

$$\left(-[\mathbb{A}]+[\mathbb{B}]\right)(\Xi) = (C), \qquad (12)$$

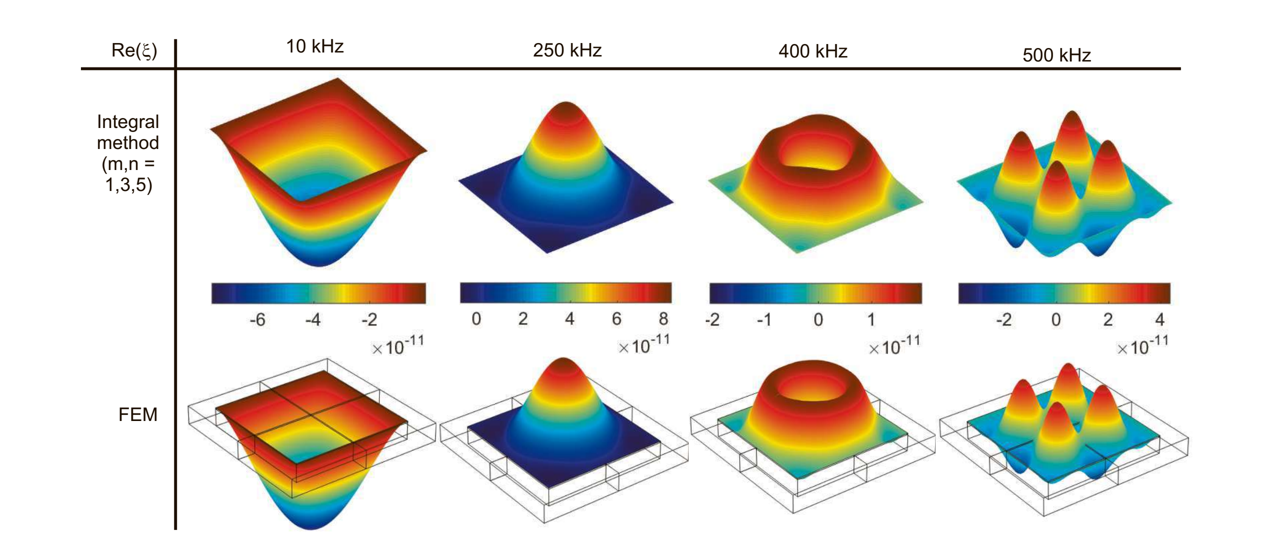

Figure 2. Real parts of the membrane displacement in metres given by the method presented herein in using first 9 eigenmodes (upper mappings) and calculated by the finite element method (lower mappings) at 10 kHz, 250 kHz, 400 kHz and 500 kHz.

where (Ξ) and (C) are, respectively, the column vectors of elements ξ_{mn} and

$$c_{mn} = -p_{inc} \iint\limits_{(-a,a)\times(-a,a)} \psi_{mn}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{13}$$

$[\mathbb{B}]$ is the diagonal matrix of elements $T(k_{mn}^2 - K_M^2)$, and where the matrix $[\mathbb{A}]$ of elements $A_{(mn),(qr)}$ is given by [after using Equations (6), (7), and (8)]

$$\begin{aligned} A_{(mn),(qr)} = {} & \frac{\rho_0\omega^2}{h_g F_v}\int_{-a}^{a}\int_{-a}^{a}\Bigg[\psi_{mn}(x,y) \\ & \cdot \int_0^a\int_0^a G(x,x_0;y,y_0)\psi_{qr}(x_0,y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0 \\ & - \frac{1}{1+\langle I_G(x,a)\rangle_x} \\ & \cdot \int_0^a\int_0^a \big\langle G(x,x_0;a,y_0)\big\rangle_x\,\psi_{qr}(x_0,y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0 \\ & \Bigg\{\left[\frac{\mathrm{j}\omega\rho_0}{8F_v Z_c h_g a}\psi_{mn}(x,y) + \partial_y\psi_{mn}(x,y)\right] \\ & \cdot \int_0^a G(x,x_0;y,a)\,\mathrm{d}x_0 \\ & + \left[\frac{\mathrm{j}\omega\rho_0}{8F_v Z_c h_g a}\psi_{mn}(x,y) + \partial_x\psi_{mn}(x,y)\right] \\ & \cdot \int_0^a G(x,a;y,y_0)\,\mathrm{d}y_0\Bigg\}\Bigg]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{14}$$

It is worthwhile commenting that the Green's function in the integrals in expressions (14) has integrand singularities, but the limits of integrations are finite. The integrand tends logarithmically to infinity as the argument u of the Green's function tends to zero, but the integral converges. In expression (7), the Green's function is differentiated with respect to x_0 and y_0 (first order derivatives) and then the integrand behaves as $(1/u)$ as $u \to 0$. Therefore, in order to ensure efficiency of the numerical integrations, which are handled in using MATLAB software, the derivatives of the Green's function in the integrands are removed on using first the reciprocity property $\partial_{x_0}G(x,x_0;y,y_0) = -\partial_x G(x,x_0;y,y_0)$ and second on integrating by part, which makes appear the spatial derivatives of $\psi_{mn}(x,y)$ in Equation (14).

3. Transducers with square membrane: results and discussions

For the case of a square membrane, in using the well-known set of orthogonal functions $\psi_{mn}(x,y) = \cos(k_{x_m}x)\cos(k_{y_n}y)/a$ [5], the elements of the column vector (C) (Eq. 13) can be then expressed as

$$c_{mn} = -p_{inc}16a(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}/(mn\pi^2). \tag{15}$$

The results for the transducer with square silicon membrane of thickness $h_d = 10\,\mu\mathrm{m}$, half-side $a = 0.5$ mm and tension $T = 945$ N/m, and with other fluid domain dimensions $h_g = 10\,\mu\mathrm{m}$, $V_c = 10^{-10}\,\mathrm{m}^3$ are presented below. The typical physical properties of air and silicon can be found for example in [4]. First, the comparison between the analytical results given by the method presented herein using first 9 eigenmodes, $m,n = 1,3,5$ and the results obtained in using the finite element method for the real part of the membrane displacement field in metres is shown in Figure 2. A good agreement can be noticed, especially at 10 kHz and 250 kHz. At 400 kHz the quasi-circular shape is not perfectly represented in using only first 9 eigenmodes and at 500 kHz the amplitude of the displacement is slightly overestimated, but these discrepancies remain small.

The pressure sensitivity of the transducer considered as a receiver is given by $\sigma = -U_0\bar{\xi}/(p_{inc}h_g)$, where $\bar{\xi}$ is the mean displacement of the membrane over the surface of the backing electrode and U_0 is the polarization voltage (herein $U_0 = 30$ V). The frequency dependence of the pressure sensitivity of the transducer mentioned above is shown in Figure 3. A good agreement between the analytical (solid lines) and numerical ("×" marks) results is obtained at the frequencies where the membrane displacement can be represented by the given number of eigenmodes (up to 400 kHz for $m,n = 1,3$ and up to 600 kHz for $m,n = 1,3,5$ approximately).

Note that the FEM solutions, against which the analytical results have been tested, are provided using the linear formulation available in the literature [8], which shows very good agreement with experimental results [9]. The use of this formulation in the modeling of electrostatic transducers is described for example in [4].

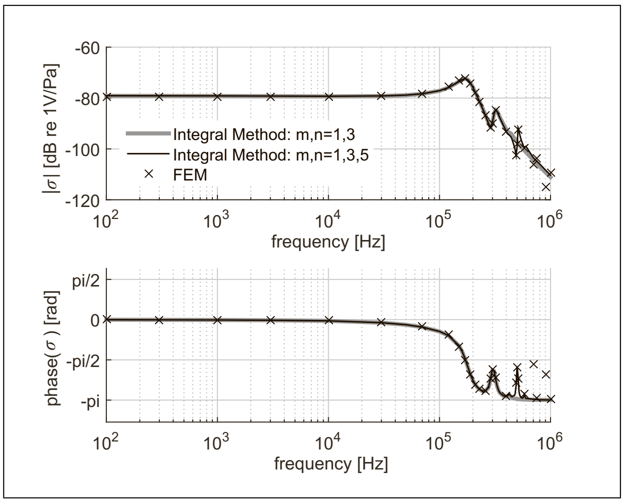

Figure 3. Magnitude (upper curves) and phase (lower curves) of the pressure sensitivity of the miniaturized receiver calculated analytically using first 4 eigenmodes (thick gray line), first 9 eigenmodes (thin black line) and calculated numerically ("×" marks).

4. Conclusion

An integral formulation for describing the behavior of the pressure field inside a thin layer of thermo-viscous fluid of finite extent, trapped between a vibrating wall and a planar boundary, has been worked out, which account for thermo-viscous boundary layer effects and coupling effects. The limitations of the standard analytic procedure whereby one expresses the acoustic field as a sum over the eigenmodes of the fluid film which are coupled with modal expressions of the membrane displacement are avoided. The alternate procedure proposed in this paper is to make use of Green's functions depending on two coordinates and to not express such Green's functions as sums over eigenfunctions. While this analytical modelling could appear somewhat cumbersome, the numerical integrations are in fact very simple and rapid to handle. The model leads to the expected results. Thus, the predictions of this model should be useful to describe phenomena in a variety of devices.

Acknowledgement

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS15/226/OHK2/3T/16.

References

[1] M. J. Herring Jensen, E. Sandermann Olsen: Virtual prototyping of condenser microphone using the finite element method for detailed electric, mechanic, and acoustic characterisation. Proceeding of Meetings on Acoustics **19** (2013) 030039.

[2] S. Barrera-Figueroa, K. Rasmussen, F. Jacobsen: Hybrid method for determining the parameters of condenser microphones from measured membrane velocities and numerical calculations. J. Acoust. Soc. Am. **126** (2009) 1788-1795.

[3] W. R. Kampinga, Y. H. Wijnant, A. de Boer: An Efficient Finite Element Model for Viscothermal Acoustics. Acta Acustica united with Acustica **97** (2011) 618-631.

[4] P. Honzík, A. Podkovskiy, S. Durand, N. Joly, M. Bruneau: Analytical and numerical modeling of an axisymmetrical electrostatic transducer with interior geometrical discontinuity. J. Acoust. Soc. Am. **134** (2013) 3573-3579.

[5] M. Bruneau, A.-M. Bruneau, P. Dupire: A Model for Rectangular Miniaturized Microphones. Acta Acustica **3** (1995) 275-282.

[6] G. Plantier, M. Bruneau: Heat conduction effects on the acoustic response of a membrane separated by a very thin air film from a backing electrode. J. Acoustique **3** (1990) 243-250.

[7] W. M. Beltman: Viscothermal wave propagation including acousto-elastic interaction. Part I: Theory. Journal of Sound and Vibration **227** (1999) 555-586.

[8] N. Joly: Finite Element Modeling of Thermoviscous Acoustics on Adapted Anisotropic Meshes: Implementation of the Particle Velocity and Temperature Variation. Acta Acustica united with Acustica **96**(1) (2010) 102-114.

[9] P. Honzík, N. Joly, S. Durand, D. Rodrigues, J.-N. Durocher, M. Bruneau: Finite Element Modelling of Acoustic Field inside Small Components: Application to an Annular Slit Terminated by an Aperture in an Infinite Screen. Metrologia **49** (2012) 32-40.