An alternative Biot's displacement formulation for porous materials

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This paper proposes an alternative displacement formulation of Biot's linear model for poroelastic materials. Its advantage is a simplification of the formalism without making any additional assumptions. The main difference between the method proposed in this paper and the original one is the choice of the generalized coordinates. In the present approach, the generalized coordinates are chosen in order to simplify the expression of the strain energy, which is expressed as the sum of two decoupled terms. Hence, new equations of motion are obtained whose elastic forces are decoupled. The simplification of the formalism is extended to Biot and Willis thought experiments, and simpler expressions of the parameters of the three Biot waves are also provided. A rigorous derivation of equivalent and limp models is then proposed. It is finally shown that, for the particular case of sound-absorbing materials, additional simplifications of the formalism can be obtained. © *2007 Acoustical Society of America.* [DOI: 10.1121/1.2734482]

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I. INTRODUCTION

The purpose of this paper is to propose an alternative formulation of Biot's theory[1,2] which models the deformation of a poroelastic solid saturated by a compressible fluid. Several types of materials can be modeled with this theory, including geomaterials and sound-absorbing materials. Even if Biot's theory is not able to model every type of porous materials (porous rocks…), it has been confirmed both theoretically by homogenization techniques[3,4] or volume averaging methods[5,6] and experimentally[7,8] for a wide range of materials. The application of Biot's theory to sound-absorbing materials takes its origin in the beginning of the 80's. The modeling of the viscous and thermal properties of air saturating a porous immobile solid (equivalent fluid) has been a wide research topic and many models have been proposed.[9–13] These models consist of introducing a complex density (respectively, compressibility) of air depending on frequency to take into account viscous (respectively, thermal) effects. Biot's theory has been the subject of many scientific papers and books[11,14–16] to which the reader can refer for more details.

In the paper published in 1956,[1] Biot represented the homogenized medium with six fields which are the three displacements of each homogenized phase (solid and fluid). This paper and formulation are called *original* in the following. The theory was reformulated in order to model inhomogeneous media.[2] This second formulation is referred to as the *modified* formulation. More recently, Atalla *et al.* proposed a *mixed* formulation of Biot's equations[17] whose generalized coordinates are the solid displacement and the interstitial pressure of the fluid. It exhibits four generalized coordinates instead of six for the displacement formulations. Another interest of this formulation is the introduction of an *in vacuo* stress tensor of the solid phase, which exhibits some advantages compared to the partial stress tensor of the solid phase used in the original formulation. Nevertheless, this formulation exhibits some drawbacks (valid only for harmonic problems, energy-related interpretation…) which prevents its use in the general case.

The Biot theory has nevertheless a main drawback which is not in the range of physics but lies in the scope of analytical or numerical methods to predict the response of a porous material while submitted to a given loading. The actual formulations often induce heavy analytical formulas and even discourage new analytical indicators. It is also well known that numerical models based on Biot's equations are quite huge and need tremendous calculations even for simple configurations. Given this context, it seems necessary to find alternative solutions; it is then natural to focus on Biot's equations first as they are the starting point of all analytical and numerical models.

In this paper, an alternative displacement formulation of Biot's model is proposed. Its advantage is to simplify the equations of the model without making additional assumptions. This simplification is only valid for a linear behavior of the material; in the case of nonlinearity, the present approach is not valid. Even if no new physical result is proposed in this paper, its originality is the simplification of the formalism. This simplification is available for both geomaterials and sound-absorbing materials. It is shown that for the latter, additional interesting simplifications can also be obtained.

Section II proposes an alternative choice of generalized coordinates simplifying equations of motion in the case of a nondissipative medium. Section III shows that classical Biot's results are simplified with the new formulation. Section IV focuses on the case of equivalent fluid and limp models. Section V deals with the generalization of the former results in formulations for dissipative porous material, and Sec. VI is devoted to the case of sound-absorbing materials.

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II. ALTERNATIVE SET OF GENERALIZED COORDINATES IN THE ABSENCE OF DISSIPATION

A. Strain energy and stress-strain relations

The Cartesian coordinates are denoted by $\{x_1,x_2,x_3\}$. The displacement of the homogenized solid (respectively, fluid) phase is designated by the components u_i^s (respectively, u_i^f) with $i=1,2,3$ or in a vector notation by $\mathbf{u}^s$ (respectively, $\mathbf{u}^f$). For all displacement fields, the derivative with respect to space is expressed with the generic notation $u_{i,j}=\partial u_i/\partial x_j$. The deformation is $\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ and the dilatation of solid and fluid phase are, respectively, $e=u_{i,i}^s$ and $\varepsilon=u_{i,i}^f$, with convention for repeated indices. The deformation tensor is denoted by $\boldsymbol{\varepsilon}$ in tensor form.

In the original paper,[1] Biot proposed use of $\mathbf{u}^s$ and $\mathbf{u}^f$ as a set of generalized coordinates and the total stress tensor was separated into two parts. The first and second parts are the stress components acting on the solid and fluid phase, respectively. The corresponding stress tensor is denoted by σ_{ij}^s (respectively, $\sigma_{ij}^f=\sigma\delta_{ij}$), where δ_{ij} denotes the Kronecker symbol. Hence, the fluid partial stress tensor is isotropic and diagonal. The stress of the fluid part is represented by σ; it is linked to the porosity ϕ and the fluid pressure p_f by the relation $\sigma=-\phi p_f$. In tensor form, the stresses are denoted by a bold symbol (e.g., $\boldsymbol{\sigma}^s$).

In 1962 paper,[2] Biot proposed to use $\mathbf{u}^s$ and $\mathbf{w}=\phi(\mathbf{u}^f-\mathbf{u}^s)$ as generalized coordinates, where $\mathbf{w}$ is the flow of the fluid relative to the solid measured in terms of volume per unit area of the bulk medium. This new choice of generalized coordinates induces a modification of conjugate variables which are the total stress tensor (noted τ_{ij}) and the fluid pressure p_f.

The strain energy of a porous elastic solid saturated by a fluid can be defined as the isothermal free energy of the fluid-solid system. In $\{\mathbf{u}^s,\mathbf{u}^f\}$ formulation, this energy W_0 reads

$$W_0=A\frac{e^2}{2}+R\frac{\varepsilon^2}{2}+2N\varepsilon_{ij}^s\varepsilon_{i,j}^s+Qe\varepsilon. \tag{1}$$

A, R, N, and Q are the constitutive coefficients of the homogenized porous medium.[18] N is the shear modulus of the skeleton. Q is a coupling coefficient between the dilatation and stress of the two phases; R may be interpreted as the bulk modulus of the air occupying a fraction ϕ of a unit volume of aggregate. The elastic coefficients A, Q, and R can be obtained by the Biot and Willis experiments[18] from K_b, the bulk modulus of the skeleton *in vacuo*, from K_s, the bulk modulus of the elastic solid from which the skeleton is made, and from K_f, the bulk modulus of the fluid in the pores. A and N are the Lamé coefficient of the solid partial stress tensor. The expression[18] of A shows a dependence on K_f. Hence, this apparent solid parameter depends on the interstitial fluid property.

In 1962 formulation, the strain energy is then written as

$$W_1=A'\frac{e^2}{2}+2N\varepsilon_{ij}^s\varepsilon_{i,j}^s-\frac{Q+R}{\phi}e\xi+\frac{R}{\phi^2}\frac{\xi^2}{2}, \tag{2}$$

with $A'=A+2Q+R$ and $\xi=-\boldsymbol{\nabla}\cdot\mathbf{w}$. It can easily be checked that W_0 and W_1 are equivalent. The stress-strain relations of the porous media are obtained from Helmholtz relations, and for each formulation the two corresponding stress tensors depend on both generalized coordinates. Biot wrote the expressions of W_0 and W_1 by way of the virtual work of surface forces. An alternative way to obtain W_1 is to substitute in (1) the expression of ε derived from the definition of $\mathbf{w}$. The expressions of W_0 and W_1 are not formally different as both are the sum of three types of terms. The first type corresponds to quadratic terms associated with the solid deformation, the second with quadratic terms related to the dilatation of the considered second generalized coordinate, and the last with coupling terms.

B. Strain decoupled formulation

This section is central in this paper. Its purpose is to propose a strain decoupled formulation. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be an adapted set of generalized coordinates. Without loss of generality, the following linear relations can be written:

$$\mathbf{u}^s=a\mathbf{u}_1+b\mathbf{u}_2,\quad \mathbf{u}^f=c\mathbf{u}_1+d\mathbf{u}_2. \tag{3}$$

The strain energy W_2 is written as

$$\begin{aligned}W_2=&\frac{e_1^2}{2}(Aa^2+Rc^2+2Qac)+\frac{e_2^2}{2}(Ab^2+Rd^2+2Qbd)\\&+e_1e_2(Aab+Rcd+Q(ad+bc))+2N(a^2\varepsilon_{ij}^1\varepsilon_{ij}^1\\&+b^2\varepsilon_{ij}^2\varepsilon_{ij}^2+2ab\varepsilon_{ij}^1\varepsilon_{ij}^2).\end{aligned}$$

For the sake of simplicity it seems natural to avoid $\varepsilon_{ij}^1\varepsilon_{ij}^2$ and $\varepsilon_{ij}^2\varepsilon_{ij}^2$ terms. This implies that $b=0$ is an appropriate choice. Hence, e_1e_2 term is avoided if $c=-(Q/R)a$ and

$$W_2=a^2\hat{A}\frac{e_1^2}{2}+d^2R\frac{e_2^2}{2}+2Na^2\varepsilon_{ij}^1\varepsilon_{ij}^1, \tag{4}$$

with $\hat{A}=(A-(Q^2/R))$. All choices of a and d are mathematically equivalent. The adequate choice is $a=1$, so that $\mathbf{u}_1=\mathbf{u}^s$ and $d=\phi^{-1}$ in order to limit the influence of porosity on the model. The new generalized coordinates are now totally determined and the strain decoupled formulation is called $\{\mathbf{u}^s,\mathbf{u}^W\}$, with

$$\mathbf{u}^W=\phi\left(\mathbf{u}^f+\frac{Q}{R}\mathbf{u}^s\right), \tag{5}$$

$$W_2=\hat{A}\frac{e^2}{2}+K_{\mathrm{eq}}\frac{\zeta^2}{2}+2N\varepsilon_{ij}^s\varepsilon_{ij}^s,\quad \zeta=\boldsymbol{\nabla}\cdot\mathbf{u}^W,\quad K_{\mathrm{eq}}=\frac{R}{\phi^2}. \tag{6}$$

K_{eq} corresponds to the compressibility of the equivalent fluid model; it is now introduced in order to condensate the expression of the equations directly from now. The stress-strain relations for $\{\mathbf{u}^s,\mathbf{u}^W\}$ formulation read

$$\hat{\sigma}_{ij}^s=2N\varepsilon_{ij}^s+\hat{A}e\delta_{ij},\quad p_f=-K_{\mathrm{eq}}\zeta. \tag{7}$$

Unlike the solid partial stress tensor σ_{ij}^s, which is a function of both solid and fluid phase displacements, $\hat{\sigma}_{ij}^s$ only depends on the motion of the solid phase; this tensor $\hat{\sigma}_{ij}^s$ is called jacketed stress tensor of the solid phase by analogy to Biot and Willis' second experiment. This tensor was proposed by Atalla *et al.*[17] and called *in vacuo* stress tensor. Hence, the strain energy is the sum of two terms (and not three as for W_0 and W_1). The stresses appearing in $\{\mathbf{u}^s,\mathbf{u}^W\}$ formulation are the jacketed stresses of the solid phase and the pressure. A first remark is that each stress is associated with its corresponding displacement then avoiding coupling terms. A second remark is that the pressure p_f can be expressed as the divergence of only $\mathbf{u}^W$ in (7). In the case of a motionless solid, one has $\mathbf{u}^W=\phi\mathbf{u}^f$. Hence, $\mathbf{u}^W$ corresponds to the average of the microscopic fluid displacement on the total volume of the porous medium.

It is also interesting to express the total stress tensor of the porous medium which defines an interesting coefficient,

$$\tau_{ij}=\hat{\sigma}_{ij}-\gamma' p_f, \quad \gamma'=\phi\left(1+\frac{Q}{R}\right). \tag{8}$$

Hence, it is important to notice that $\hat{\boldsymbol{\sigma}}$ corresponds to the effective stress tensor $\boldsymbol{\tau}'$ defined by Biot in a 1962 paper if and only if $\gamma'=1$. This coefficient plays a central role in the following and in particular for the expression of kinetic energies which is now considered.

C. Kinetic energy and equations of motion

The preceding subsection was only concerned with strain energies. In order to obtain the equations of motion, it is necessary to also express the kinetic energies. Biot's definitions of densities are

$$\rho_1=(1-\phi)\rho_m, \quad \rho_2=\phi\rho_f, \quad \rho_{12}=-\phi\rho_f(\alpha_\infty-1), \tag{9a}$$

$$\rho_{11}=\rho_1-\rho_{12}, \quad \rho_{22}=\rho_2-\rho_{12}, \quad \rho_{\mathrm{eq}}=\frac{\rho_{22}}{\phi^2}. \tag{9b}$$

ρ_m is the density of the matter constituting the solid phase and ρ_f is the density of the fluid saturating the pores. ρ_{12} is an inertial coupling coefficient linked to the geometric tortuosity α_∞. ρ_{eq} corresponds to the density of the equivalent fluid model and, analogously to K_{eq}, is now defined in order to simplify the expressions.

The equations of motion in $\{\mathbf{u}^s,\mathbf{u}^W\}$ formulation are now obtained. The first step consists of substituting in T the expression of $\mathbf{u}^f$ as a linear combination of $\mathbf{u}^s$ and $\mathbf{u}^W$. One has

$$T_2=\frac{\rho_s}{2}\dot{\mathbf{u}}^{s2}+\frac{\rho_{\mathrm{eq}}}{2}\dot{\mathbf{u}}^{W2}+\rho_{\mathrm{eq}}\gamma\dot{\mathbf{u}}^s\dot{\mathbf{u}}^W, \tag{10}$$

with

$$\gamma=\phi\left(\frac{\rho_{12}}{\rho_{22}}-\frac{Q}{R}\right), \quad \rho_s=\rho_1+\rho_2\left(\frac{Q}{R}\right)^2-\rho_{12}\frac{\gamma'^2}{\phi^2}. \tag{11}$$

The equations of motion in the $\{\mathbf{u}^s,\mathbf{u}^W\}$ formulation read

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\sigma}}(\mathbf{u}^s)=\rho_s\ddot{\mathbf{u}}^s+\rho_{\mathrm{eq}}\gamma\ddot{\mathbf{u}}^W, \tag{12a}$$

$$K_{\mathrm{eq}}\boldsymbol{\nabla}\zeta=\rho_{\mathrm{eq}}\gamma\ddot{\mathbf{u}}^s+\rho_{\mathrm{eq}}\ddot{\mathbf{u}}^W. \tag{12b}$$

These equations are equivalent to those proposed by Biot. Unlike the original ones, there is no stress coupling terms in them, and each stress tensor is a function of only the corresponding displacement. The symmetry is also preserved for inertial terms. It is shown in the following sections that the classical results of Biot's theory can easily be retrieved from the present formulation with the advantage of simpler expressions.

III. ADAPTATION OF CLASSICAL BIOT'S RESULTS

A. Biot and Willis experiments

Biot and Willis[18] presented three thought experiments which provide expressions for the elastic coefficient appearing in Biot's original model: A, N Q, and R. In the case of the strain decoupled formulation it is shown that the thought experiments provide the three elastic coefficients for the model $\hat{A}$, N, and K_{eq} and an expression of γ'. Biot and Willis experiments assume quasistatic deformation. In a recent contribution, Lafarge[16] shows that this assumption is not restrictive and that these experiments can be extended to harmonic excitations.

The first thought experiment is a measure of the shear modulus N of the material and consequently the shear modulus of the frame, since the fluid does not contribute to the shear force.

In the second thought experiment (called jacketed experiment), the material is surrounded by a flexible jacket that is subjected to a pressure p_{jac}. The fluid inside the jacket remains at the ambient pressure. It follows that $p_f=0$ and $\hat{\sigma}_{ij}=-p_{\mathrm{jac}}$. The deformation of the solid phase is denoted by e_{jac}. The stress-strain relation (7) implies that

$$-p_{\mathrm{jac}}=\left(\hat{A}+\frac{2N}{3}\right)e_{\mathrm{jac}}. \tag{13}$$

This last relation must be linked to the definition of the bulk modulus K_b of the frame at constant pressure in the air $K_b=-(p_{\mathrm{jac}}/e_{\mathrm{jac}})$, and one obtains

$$\hat{A}=K_b-\frac{2N}{3}. \tag{14}$$

The last thought experiment is called an unjacketed experiment and provides two additional equations. The material is subjected to an increase of pressure p_u in the fluid inducing a total stress equal to $\tau_{ij}=-p_u\delta_{ij}$. The divergence of the $\mathbf{u}^s$ (respectively, $\mathbf{u}^f$ and $\mathbf{u}^W$) is called e_u (respectively, ε_u and ζ_u). Concerning this experiment, Biot introduced two coefficients,

$$K_f=-\frac{p_u}{\varepsilon_u}, \quad K_s=-\frac{p_u}{e_u}. \tag{15}$$

The stress-strain relations (7) and (8) are now expressed

$$-p_u=K_be_u-\gamma' p_u, \quad K_{\mathrm{eq}}=-\frac{p_u}{\zeta_u}. \tag{16}$$

The first equation of (16) enables an expression of γ' as a function of K_s and K_b,

$$\gamma'=1-\frac{K_b}{K_s}. \tag{17}$$

This last result is linked to the second equation of (16) and the expression for K_{eq} is provided,

$$K_{\rm eq} = \frac{K_f}{\phi + (1-\phi)\dfrac{K_f}{K_s} - \dfrac{K_b K_f}{K_s^2}}. \quad (18)$$

Biot and Willis results for A, N, Q, and R can rigorously be obtained from the expressions of $\hat{A}$, N, γ', and $K_{\rm eq}$. Nevertheless, the expressions of the latter are simpler. In particular, it is interesting to notice that $\hat{A}$ does not depend on K_f, unlike the constitutive coefficient A of Biot's original formulation.

A second and fundamental remark is that γ' is independent of the compressibility of the fluid through (17), and it is possible to express $\mathbf{u}^W$ as

$$\mathbf{u}^W = \underbrace{\phi \mathbf{u}^f + (1-\phi)\mathbf{u}^s}_{\mathbf{u}^t} - \frac{K_b}{K_s}\mathbf{u}^s. \quad (19)$$

$\mathbf{u}^t$ is called the total displacement of the porous material. It is shown here that $\mathbf{u}^W$ is independent of porosity. It is the *a posteriori* justification of the particular choice $d=\phi^{-1}$ considered in the preceding section.

B. Wave numbers of the Biot's waves

This section deals with the rewriting of the wave numbers of the three Biot's waves. The methodology is the same that the one proposed in Ref. 11. The two compressional waves are first studied. Two scalar potentials φ^s and φ^W are defined for the compressional waves. Hence, equations of motion of the strain decoupled formulation (12) are written as

$$-\omega^2[\boldsymbol{\rho}]\begin{Bmatrix} \varphi^s \\ \varphi^W \end{Bmatrix} = [\mathbf{K}]\nabla^2 \begin{Bmatrix} \varphi^s \\ \varphi^W \end{Bmatrix}, \quad (20)$$

where $[\boldsymbol{\rho}]$ and $[\boldsymbol{K}]$ are, respectively,

$$[\boldsymbol{\rho}] = \begin{bmatrix} \rho_s & \rho_{\rm eq}\gamma \\ \rho_{\rm eq}\gamma & \rho_{\rm eq} \end{bmatrix}, \quad [\mathbf{K}] = \begin{bmatrix} \hat{P} & 0 \\ 0 & K_{\rm eq} \end{bmatrix}, \quad (21)$$

with $\hat{P}=\hat{A}+2N$. Let δ_1^2 and δ_2^2 be the eigenvalues of the problem associated with matrices $[\boldsymbol{\rho}]$ and $[\mathbf{K}]$. An elementary algebraic calculation gives

$$\delta_i^2 = \frac{(\delta_{s2}^2 + \delta_{\rm eq}^2) \pm \sqrt{(\delta_{s2}^2 + \delta_{\rm eq}^2)^2 - 4\delta_{\rm eq}^2\delta_{s1}^2}}{2}, \quad (22)$$

with

$$\delta_{\rm eq} = \omega\sqrt{\frac{\rho_{\rm eq}}{K_{\rm eq}}}, \quad \delta_{s1} = \omega\sqrt{\frac{\rho}{\hat{P}}}, \quad \delta_{s2} = \omega\sqrt{\frac{\rho_s}{\hat{P}}},$$

$$\rho = \rho_s - \gamma^2\rho_{\rm eq}. \quad (23)$$

These expressions are equivalent to the classical expressions of these two wave numbers which can be found in Ref. 11. It is quite evident that the proposed expressions are more condensed than the classical ones. The main reason is that definitions (22) are expressed only through the three intrinsic wave numbers defined in (23). This analytical simplification was used particularly by Dazel and Pilon[19] in order to define new types of decoupling criteria between the two compressional waves.

The following symmetric relations exist between the wave numbers:

$$\delta_1^2\delta_2^2 = \delta_{s1}^2\delta_{\rm eq}^2, \quad \delta_1^2 + \delta_2^2 = \delta_{s2}^2 + \delta_{\rm eq}^2. \quad (24)$$

$\delta_{\rm eq}$ is the wave number of the equivalent fluid model i.e., when the solid phase is immobile; more details will be found in Sec. IV. Symmetrically to the equivalent fluid model which assumes that $\mathbf{u}^s=\mathbf{0}$, an equivalent solid model can be considered, for which it is postulated that $\mathbf{u}^W=\mathbf{0}$. In this model only one compressional wave propagates whose wave number is δ_{s2}. Even if there is a perfect mathematical symmetry between these two cases, the first one is physically realistic (and has been often used in the past) while the second is not. δ_{s1} is the wave number of the wave propagating in the solid if the porous medium is in vacuum (and not saturated by air).

The eigenvectors are determined by the ratio μ_i^W of the u^W component on the u^s one. Two possible and equivalent expressions for this ratio are

$$\mu_i^W = \gamma\frac{(\delta_i^2 - \delta_{s2}^2)}{\delta_{s2}^2 - \delta_{s1}^2} = \gamma\frac{\delta_{\rm eq}^2}{\delta_i^2 - \delta_{\rm eq}^2}. \quad (25)$$

As symmetric relations (24) were obtained for the wave numbers, orthogonality relations can be obtained on μ_i^W,

$$\hat{P} + K_{\rm eq}\mu_1^W\mu_2^W = 0, \quad (26a)$$

$$\rho_s + \rho_{\rm eq}\gamma(\mu_1^W + \mu_2^W) + \rho_{\rm eq}\mu_1^W\mu_2^W = 0. \quad (26b)$$

It is also interesting to introduce the following ratios:

$$\mu_i' = \frac{\mu_i^W}{\mu_i^W - \mu_j^W} = \frac{\delta_j^2 - \delta_{\rm eq}^2}{\delta_j^2 - \delta_i^2}, \quad \text{with} \quad (i,j) \in \{1,2\}. \quad (27)$$

The shear wave is now considered by using a vector potential,

$$\mathbf{u}^s = \boldsymbol{\nabla} \wedge \Psi \quad \text{so} \quad \mathbf{u}^W = \mu_3^W \boldsymbol{\nabla} \wedge \Psi. \quad (28)$$

Substituting these expressions in the motion Eqs. (12), one obtains

$$\delta_3 = \omega\sqrt{\frac{\rho}{N}} \quad \text{and} \quad \mu_3^W = -\gamma. \quad (29)$$

IV. EQUIVALENT FLUID AND LIMP MODELS

The equivalent fluid model corresponds to a motionless solid phase ($\mathbf{u}^s=\mathbf{0}$). Equation (12b) becomes

$$K_{\rm eq}\nabla^2\mathbf{u}^W = \rho_{\rm eq}\ddot{\mathbf{u}}^W. \quad (30)$$

It is straightforward to find that the wave number of the equivalent fluid model is $\delta_{\rm eq}$. The characteristic impedance of the equivalent fluid in the strain decoupled formulation is defined as

$$Z_{\rm eq} = \frac{p_f}{\|\mathbf{v}^W\|} = \sqrt{\rho_{\rm eq}K_{\rm eq}}. \quad (31)$$

Let us now consider the limp model which also exhibits one compressional wave. Unlike the equivalent fluid model, the solid is not motionless and this model takes into account the inertia of the fluid phase. It is associated with materials whose rigidity is negligible (light mineral wools, cotton…). The jacketed strain energy of the solid phase is negligible compared to those of the other mechanisms of the propagation so that $\hat{A} \simeq 0 \simeq N$. Hence, the compressional term $\nabla \cdot \hat{\sigma}$ can be neglected in (12a), which gives a relation between $\ddot{\mathbf{u}}^s$ and $\ddot{\mathbf{u}}^W$,

$$\rho_s \ddot{\mathbf{u}}^s = -\rho_{\text{eq}} \gamma \ddot{\mathbf{u}}^W. \tag{32}$$

This relation is now inserted in (12b) and a propagation equation on $\mathbf{u}^W$ is obtained as

$$K_{\text{eq}} \nabla \zeta = \rho_{\text{eq}} \left(1 - \frac{\rho_{\text{eq}} \gamma^2}{\rho_s} \right) \ddot{\mathbf{u}}^W. \tag{33}$$

The limp model is a one-compression wave model whose difference with the equivalent fluid is the definition of the density,

$$\rho_{\text{limp}} = \rho_{\text{eq}} \left(\frac{\delta_{s1}}{\delta_{s2}} \right)^2. \tag{34}$$

The wave number of the limp model can now be expressed as a function of the three intrinsic wave numbers (23) of the porous medium,

$$\delta_{\text{limp}} = \omega \sqrt{\frac{\rho_{\text{limp}}}{K_{\text{eq}}}} = \delta_{\text{eq}} \frac{\delta_{s1}}{\delta_{s2}}. \tag{35}$$

The characteristic impedance of the limp model is

$$Z_{\text{limp}} = \frac{p_f}{\|\mathbf{v}^W\|} = \sqrt{\rho_{\text{limp}} K_{\text{eq}}}. \tag{36}$$

Two one-compressional wave models were presented in this section. It has been shown that the $\{\mathbf{u}^s, \mathbf{u}^W\}$ formulation is well fitted to these two types of model.

V. STRAIN DECOUPLED FORMULATION WITH DISSIPATION

This section deals with the introduction of dissipative effects in the formulation. It is shown that the symmetry of Eq. (12) is preserved even if dissipation is considered. The dissipation is taken into account for harmonic excitation by modifying the constitutive and inertial coefficients of the model.

Viscous dissipation was introduced by Biot in 1956,[1] with the assumption that the flow of the fluid relative to the solid through the pores is of Poiseuille type. In order to integrate this dissipation in the Lagrangian formulation, a dissipation function D was defined as a homogeneous quadratic form with the six generalized velocities. This function is first rewritten in term of the generalized coordinates of our proposed approach,

$$D = \frac{\sigma \phi^2 G}{2} |\dot{\mathbf{u}}^f - \dot{\mathbf{u}}^s|^2 = \frac{\sigma G}{2} |\dot{\mathbf{u}}^W - \gamma' \dot{\mathbf{u}}^s|^2, \tag{37}$$

where σ is the flow resistivity of the porous sample and G is a nondimensional correction function. This function is useful to represent the variation of apparent viscosity versus frequency. This function is first assumed to be a constant, and its dependence versus frequency is considered at the end of this section. The Euler Lagrange equations read

$$\nabla \cdot \hat{\boldsymbol{\sigma}}(\mathbf{u}^s) = \rho_s \ddot{\mathbf{u}}^s + \rho_{\text{eq}} \gamma \ddot{\mathbf{u}}^W + \sigma G (\gamma'^2 \dot{\mathbf{u}}^s - \gamma' \dot{\mathbf{u}}^W), \tag{38a}$$

$$K_{\text{eq}} \nabla \zeta = \rho_{\text{eq}} \gamma \ddot{\mathbf{u}}^s + \rho_{\text{eq}} \ddot{\mathbf{u}}^W + \sigma G (\dot{\mathbf{u}}^W - \gamma' \dot{\mathbf{u}}^s). \tag{38b}$$

The right-hand sides of these two equations are rewritten by using (11), and one obtains

$$\nabla \cdot \hat{\boldsymbol{\sigma}}(\mathbf{u}^s) = \left[\rho_1 + \rho_2 \left(\frac{Q}{R} \right)^2 \right] \ddot{\mathbf{u}}^s + \frac{\rho_2 (\phi - \gamma')}{\phi^2} \ddot{\mathbf{u}}^W - \frac{\gamma'^2}{\phi^2} V(\mathbf{u}^s) + \frac{\gamma'}{\phi^2} V(\mathbf{u}^W), \tag{39a}$$

$$K_{\text{eq}} \nabla \zeta = \frac{\rho_2}{\phi^2} [\phi - \gamma'] \ddot{\mathbf{u}}^s + \frac{\rho_2}{\phi^2} \ddot{\mathbf{u}}^W + \frac{\gamma'}{\phi^2} V(\mathbf{u}^s) - \frac{1}{\phi^2} V(\mathbf{u}^W), \tag{39b}$$

with the time differential operator V defined by the functional relation,

$$V(\mathbf{u}) = \rho_{12} \ddot{\mathbf{u}} - \phi^2 \sigma G \dot{\mathbf{u}}. \tag{40}$$

The function G is actually frequency dependent.[9–13] In the case of harmonic excitation at circular frequency ω, the complex notation is used ($e^{j\omega t}$ dependence). One obtains

$$V(\mathbf{u}) = -\omega^2 \tilde{\rho}_{12} \mathbf{u}, \quad \tilde{\rho}_{12} = \rho_{12} - \frac{\phi^2 \sigma G(\omega)}{j\omega}. \tag{41}$$

It is then possible to define the complex dissipative extensions of the coefficients introduced in the preceding sections,

$$\tilde{\rho}_{22} = \rho_2 - \tilde{\rho}_{12}, \quad \tilde{\rho}_{11} = \rho_1 - \tilde{\rho}_{12}, \quad \tilde{\alpha} = \frac{\tilde{\rho}_{22}}{\rho_2}, \quad \tilde{\gamma} = \frac{\phi}{\tilde{\alpha}} - \gamma'. \tag{42}$$

The frequency equations associated with the viscous dissipating problem are

$$\nabla \cdot \hat{\boldsymbol{\sigma}}(\mathbf{u}^s) = -\omega^2 \tilde{\rho}_s \mathbf{u}^s - \omega^2 \tilde{\rho}_{\text{eq}} \tilde{\gamma} \mathbf{u}^W, \tag{43a}$$

$$K_{\text{eq}} \nabla \zeta = -\omega^2 \tilde{\rho}_{\text{eq}} \tilde{\gamma} \mathbf{u}^s - \omega^2 \tilde{\rho}_{\text{eq}} \mathbf{u}^W. \tag{43b}$$

Various models of viscosity of air saturating an immobile porous solid [i.e., of $G(\omega)$] have been proposed in the past[9,10] which can be used for this formulation without restriction. Even if Eqs. (43a) is expressed for harmonic motions, it can be noticed that it directly corresponds to (12) with the time-independent coefficients replaced by frequency-dependent ones.

The structural dissipation in the skeleton is taken into account by modifying the elastic coefficients of the jacketed stress tensor. As a temporal dependence is assumed, complex frequency-dependent extensions $\widetilde{K}_s$, $\widetilde{K}_b$, and $\widetilde{N}$ can be used in the experiments instead of the constant and real parameters used in Sec. III A. Hence, frequency-dependent coefficients for $\hat{A}$ and N can be used in order to take into account the structural dissipation. This exhibits an advantage of the use of this tensor instead of the partial stress tensor of the solid phase whose parameters depends on both structural and thermal dissipation.

The thermal effects are taken in account by modifying K_f, which is now

$$\widetilde{K}_f = \frac{K_a}{\beta(\omega)}, \tag{44}$$

with K_a the adiabatic compressibility coefficient of air and $\beta(\omega)$ the thermal dynamic susceptibility. This modification acts only on K_{eq}. Like the viscous function G, various models have been proposed in order to explicit this function; the reader can refer to these models[11,13] which can be used for $\{\mathbf{u}^s,\mathbf{u}^W\}$ formulation without restriction.

VI. ADVANTAGE OF $\{\mathbf{u}^s,\mathbf{u}^W\}$ FORMULATION FOR SOUND-ABSORBING MATERIALS

This section is devoted to porous materials with a very stiff skeleton. Usual sound-absorbing materials are in this category. This assumption induces additional simplifications which are now detailed.

A. Introduction of the total displacement

The high stiffness of the solid matter means that

$$\left|\frac{\widetilde{K}_b}{\widetilde{K}_s}\right| \ll 1, \quad \left|\frac{\widetilde{K}_f}{\widetilde{K}_s}\right| \ll 1. \tag{45}$$

This assumption implies simplifications in both expressions of γ' (17) and $\mathbf{u}^W$ (19),

$$\gamma' \simeq 1, \quad \mathbf{u}^W \simeq \mathbf{u}^t. \tag{46}$$

Hence, $\mathbf{u}^W$ corresponds to the total displacement. This is an interesting result: first, it gives a direct physical interpretation of $\mathbf{u}^W$ and second, it greatly simplifies the continuity relations. In the following part of the paper, and in order to indicate that the approximation (45) is considered, all the W superscripts are replaced by t superscripts denoting the total displacement.

It is also possible to simplify $\widetilde{R}$ and $\widetilde{K}_{\text{eq}}$ in (18),

$$\widetilde{R} = \phi \widetilde{K}_f, \quad \widetilde{K}_{\text{eq}} = \frac{\widetilde{K}_f}{\phi}. \tag{47}$$

The continuity relations are now considered. The normal of the interface between the porous medium and the other media is noted $\mathbf{n}$ and any tangential vector to the connecting surface is noted $\mathbf{t}$.

The coupling with an elastic medium (superscript e) involves

$$\mathbf{u}^e \cdot \mathbf{n} = \mathbf{u}^t \cdot \mathbf{n}, \quad \mathbf{u}^e \cdot \mathbf{t} = \mathbf{u}^s \cdot \mathbf{t}, \tag{48a}$$

$$\boldsymbol{\sigma}^e \cdot \mathbf{n} = (\hat{\boldsymbol{\sigma}}^s \cdot \mathbf{n} - p_f \mathbf{n}), \quad \boldsymbol{\sigma}^e \cdot \mathbf{t} = \hat{\boldsymbol{\sigma}}^s \cdot \mathbf{t}. \tag{48b}$$

Concerning the interface with a fluid (superscript a) medium, the continuity relations are

$$\mathbf{u}^a \cdot \mathbf{n} = \mathbf{u}^t \cdot \mathbf{n}, \tag{48c}$$

$$p_a = p_f, \quad \hat{\boldsymbol{\sigma}}^s \cdot \mathbf{n} = \mathbf{0}. \tag{48d}$$

The new formulation is also interesting for the interface between two porous media,

$$\mathbf{u}_1^s = \mathbf{u}_2^s, \quad \mathbf{u}_1^t \cdot \mathbf{n} = \mathbf{u}_2^t \cdot \mathbf{n}.$$

$$p_{f,1} = p_{f,2}, \quad \hat{\boldsymbol{\sigma}}_1^s = \hat{\boldsymbol{\sigma}}_2^s. \tag{48e}$$

It can be seen from (48) that the use of $\mathbf{u}^s$ and $\mathbf{u}^t$ as general coordinates is well adapted to describe the continuity relations between two porous media. This simplification concerns both displacements and associated stresses.

B. Surface impedance of a porous material

This section is devoted to a new expression of the normal incidence surface impedance of a porous layer bonded on a rigid impervious wall. This example is inspired by an application presented in Ref. 11 (Sec. 6.6, p. 138). With the proposed formulation, the simplification of the boundary conditions allows a simplification of the final expression of the impedance.

At the surface of the porous material, three continuity conditions (48) need to be written. The first (respectively, second) one is the continuity of the pressure (respectively, total displacement). The last one is the nullity of the jacketed normal stress. The proposed set of fields is naturally adapted to these boundary conditions, while the classical $\{\mathbf{u}^s,\mathbf{u}^f\}$ implies mixture laws to obtain the total stress tensor of the porous and the total normal displacement. Hence, a determinant is obtained and the final expression of the impedance reads

$$Z = \frac{\widetilde{K}_{\text{eq}}}{j\omega} \times \frac{1}{\dfrac{\mu_2'}{\delta_2}\tan(\delta_2 l) + \dfrac{\mu_1'}{\delta_1}\tan(\delta_1 l)}, \tag{49}$$

with μ_i' defined in Eq. (27). This expression is simpler and equivalent to the classical one provided in Ref. 11. The Appendix presents alternative and simpler expressions for the reflection and transmission coefficient. A numerical example is provided in Fig. 1, where expression (49) of the normal incidence surface impedance is compared with the result provided by an industrial software MAINE 3A©,[20] developed by CTTM (based on transfer matrix method methodology for porous material.[21]) The application case is based on the example of Ref. 11. The parameters of the considered porous material are given in table 6.1 of the previous reference and the thickness of the porous layer is 5.6 cm. Figure 1 shows the perfect agreement between the original and the proposed approach.

VII. CONCLUSION

A new displacement formulation of Biot's linear equations of poroelasticity has been proposed in this paper. It

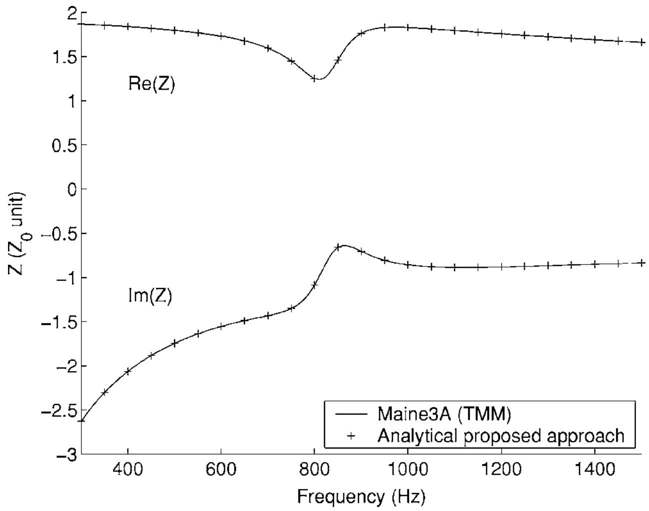

FIG. 1. Normalized surface impedance; comparison between the analytical proposed approach (+) and the numerical Maine 3A solution (continuous).

allows for a simpler expression of the different parameters of Biot's model. It is based on a choice of generalized coordinates which allows a decoupling of the strain energy of the porous medium. It is then possible to avoid the stress coupling terms in the equations of motion. The two corresponding generalized coordinates are the solid displacement and the apparent displacement for the pressure of the fluid phase taking into account the motion of the solid phase. They are associated with conjugated stresses which are the jacketed stress tensor of the solid phase and the pressure. It has also been shown that the classical Biot's results are naturally transposed to this formulation. The expressions of the constitutive coefficients, wave numbers, are equivalent and simpler in the case of the proposed approach. This formulation is also well fitted to the definition of equivalent fluid and limp model. It has been shown that the dissipation can be introduced without losing the symmetry of the problem, and the different mechanisms of dissipation can be taken into account separately. Additional simplifications were also obtained for sound-absorbing materials. Classical acoustic indicators as surface impedance, transmission, and reflection coefficients (both presented in the Appendix) have then been rewritten in a simpler form, and the expressions have been validated by a comparison to a transfer matrix method code.

The introduction of this formulation is a first step towards the simplification of numerical methods for poroelastic materials. It has been seen that for analytical results, the use of such a formulation presents great advantages compared to the classical ones. A detailed study of the discretization of these formulation is a natural perspective of this work. Another perspective of this work is the extension of the formalism to the case of inhomogeneous porous materials.

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APPENDIX: REFLECTION AND TRANSMISSION COEFFICIENTS

In this Appendix are presented two simplified expressions of the reflection and transmission coefficient of a porous media of thickness l. The considered porous media is laterally infinite and is exited by a normal incidence plane wave. The transmission (respectively, reflection) coefficient is defined as the ratio of the transmitted (respectively, reflected) pressure over the incident one as presented in Ref. 22.

The expressions of these coefficients read

$$T=\frac{-2i\xi\left[\mu_2'\delta_1'\vartheta_1+\mu_1'\delta_2'\vartheta_2\right]}{\vartheta_1\vartheta_2\left[\delta_1'\delta_2'\xi^2+\dfrac{\mu_1'^2\delta_2'^2+\mu_2'^2\delta_1'^2}{\delta_1'\delta_2'}\right]+2\mu_1'\mu_2'[1-\pi_2\pi_1]-2i\xi\left[\mu_1'\delta_2'\pi_1\vartheta_2+\mu_2'\delta_1'\pi_2\vartheta_1\right]}, \tag{A1}$$

$$R=\frac{\vartheta_1\vartheta_2\left[\delta_1'\delta_2'\xi^2-\dfrac{\mu_1'^2\delta_2'^2+\mu_2'^2\delta_1'^2}{\delta_1'\delta_2'}\right]-2\mu_1'\mu_2'[1-\pi_1\pi_2]}{\vartheta_1\vartheta_2\left[\delta_1'\delta_2'\xi^2+\dfrac{\mu_1'^2\delta_2'^2+\mu_2'^2\delta_1'^2}{\delta_1'\delta_2'}\right]+2\mu_1'\mu_2'[1-\pi_2\pi_1]-2i\xi[\mu^{1}_{\prime}\delta_2'\pi_1\vartheta_2+\mu_2'\delta_1'\pi_2\vartheta_1]}, \tag{A2}$$

with the following notations: $\xi=\widetilde{K}_{\mathrm{eq}}/K_0$, $\delta_i'=\delta_i/\delta_0$, and $\pi_i=\cos(\delta_i l)$, $\upsilon_i=\sin(\delta_i l)$. The proposed expressions, available on the overall spectrum, of the reflection and transmission coefficients have been compared to the high-frequency expressions given by Fellah *et al.*[22] in order to point out the simplifications induced by our proposed formalism.

[1] M. Biot, “Theory of propagation of elastic waves in a fluid-filled-saturated porous solid,” J. Acoust. Soc. Am. **28**, 168–191 (1956).
[2] M. Biot, “Mechanics of deformation and acoustic propagation in porous media,” J. Appl. Phys. **33**, 4, 1482–1498 (1962).
[3] R. Burridge and J. Keller, “Poroelasticity equations derived from microstructure,” J. Acoust. Soc. Am. **70**, 1140–1146 (1981).
[4] J. Auriault, L. Borne, and R. Chambon, “Dynamics of porous saturated media, checking of the generalized law of Darcy,” J. Acoust. Soc. Am. **77**, 1641–1650 (1985).
[5] S. Pride, A. Gangi, and F. Morgan, “Deriving the equations of motion for porous isotropic media,” J. Acoust. Soc. Am. **92**, 3278–3290 (1992).
[6] S. Pride and J. Berryman, “Connecting theory to experiments in poroelasticity,” J. Mech. Phys. Solids **40**, 719–747 (1998).
[7] T. Plona, “Observation of a second bulk compressional wave in a porous medium at ultrasonic frequencies,” Appl. Phys. Lett. **36**, 259–261 (1980).
[8] J. Berryman, “Confirmation of Biot’s theory,” Appl. Phys. Lett. **37**, 382–384 (1980).
[9] D. Johnson, J. Koplik, and R. Dashen, “Theory of dynamic permeability and tortuosity in fluid-saturated porous media,” J. Fluid Mech. **176**, 379–402 (1987).
[10] S. Pride, F. Morgan, and A. Gangi, “Drag forces of porous media acoustics,” Phys. Rev. B **47**, 4964–4975 (1993).
[11] J. Allard, *Propagation of Sound in Porous Media, Modeling Sound Absorbing Materials* (New York, 1993).
[12] Y. Champoux and J. Allard, “Dynamic tortuosity and bulk modulus in air-saturated porous media,” J. Appl. Phys. **70**, 1975–1979 (1991).
[13] D. Lafarge, P. Lemarinier, J. Allard, and V. Tarnow, “Dynamic compressibility of air in porous structures at audible frequencies,” J. Acoust. Soc. Am. **102**, 1995–2006 (1997).
[14] O. Coussy, *Mechanics of Porous Continua* (Wiley, New York, 1995).
[15] J. Carcione, “Wave propagation in anisotropic, saturated porous media:Plane-wave theory and numerical simulation,” J. Acoust. Soc. Am. **99**, 2655–2666 (1996).
[16] D. Lafarge, *Porous Materials, Propagation Models in Acoustics and Materials* (Hermes, Paris, in French) (2006).
[17] N. Atalla, R. Panneton, and P. Debergue, “A mixed displacement-pressure formulation for poroelastic materials”J. Acoust. Soc. Am. **104**, 1444–1452 (1998).
[18] M. Biot and D. Willis, “The elastic coefficients of the theory of consolidation,” J. Appl. Mech. **24**, 179–191 (1957).
[19] O. Dazel and D. Pilon, “Wave decoupling for porous media,” in *French Congress of Mechanics* (in French 2005).
[20] http://www.cttm-lemans.com/francais/acoustique/maine3a/sitemaine3a.html, *MAINE 3A*, CTTM - Le Mans. Viewed 5/7/2007.
[21] B. Brouard, D. Lafarge, and J. Allard, “A general method of modelling sound propagation in layered media,” J. Sound Vib. **183**, 129–42 (1995).
[22] Z. Fellah, J. Chapelon, S. Berger, W. Lauriks, and C. Depollier, “Ultrasonic wave propagation in human cancelous bone: Application of Biot theory,” J. Acoust. Soc. Am. **116**, 61–73 (2004).