Ray-wave correspondence in bent waveguides

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Abstract

The present paper is concerned with wave propagation at high frequency in bent waveguides. The multimodal formalism proposed in earlier papers by the authors is shown to be suitable for investigating very high frequency problems. Then, following works made by Luna-Acosta et al. (G. Luna-Acosta, J.A. Méndez-Bermúdez, P. Seba, K.N. Pichugin, Classical versus quantum structure of the scattering probability matrix: chaotic waveguides, Phys. Rev. E 65 (2002) 046605; J.A. Méndez-Bermúdez, G. Luna-Acosta, P. Seba, K.N. Pichugin, Understanding quantum scattering properties in terms of purely classical dynamics: two-dimensional open chaotic billiards, Phys. Rev. E 66 (2002) 046207) for open quantum billiards, the ray-wave correspondence of the scattering matrix ($S$ matrix) is studied by first constructing a $S$ matrix and then comparing its structure with the structure of the wave-$S$ matrix that is obtained with the exact multimodal formalism, for different geometries of curved waveguides. A great similarity between these two matrices is observed and it is shown that the scattering matrix constructed only by counting rays allows us to predict and understand numbers of the wave scattering properties in waveguides.

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1. Introduction

The present study is concerned with wave propagation at high frequency in bent waveguides, owing to the multimodal formalism proposed recently by the authors [1,2]. The formulation of the Euler equations in terms of the components of the pressure and velocity in the waveguide projected on the basis of the local transverse modes leads to infinite first-order differential equations, which can be numerically integrated after truncation at a finite, sufficient, number of modes [3]. Then the acoustic field or, e.g., the scattering properties can easily be calculated
for low and median frequency, up to a few tens of propagative modes. But when the frequency is increased so that a
large number of modes must be taken into account, the numerical integration of the multimodal equations becomes
difficult to achieve. However, in bends – when the cross-section and the curvature of the waveguide are constant – the
invariance along the axis of the coefficients in the multimodal equations allows algebraic solutions for the acoustic
field and scattering matrix to be carried out, avoiding thus the numerical integration process. Since the algebraic
solutions involve only simple operations – inversion, eigenvalues extraction – on $N \times N$ matrices, where $N$ is
the total number of modes, a large number of modes may then be taken into account and, as a consequence, the
acoustical properties at high frequency may be investigated.

The first aim of this paper is thus to evaluate and discuss the suitability of the multimodal formulation for
investigating high frequency problems. The algebraic solution for the pressure in a bend is carried out in Section 2
and an example of a high frequency acoustic field calculation – the synthesis of a ray by a cluster of modes – is
given and discussed. The calculation of the scattering matrix has already been presented in previous works [1,2].
Section 3 of this paper is thus limited to a brief recall on the scattering matrix and its properties, then a validation
of this algebraic formulation at high frequency is given, showing almost perfect energy conservation.

In recent papers on the quantum-classical correspondence of open chaotic billiards, Luna-Acosta et al. [4] and
Méndez-Bermúdez et al. [5] define the purely classical counterpart of the scattering probability matrix (SPM) $(S_{\text{cl}})^2$
of the quantum scattering matrix $S$ for two-dimensional waveguides and compare the structures of the quantum and
classical SPM. For a moderate number of modes ($\sim 10 - 20$) and in one case for a large number of modes (200)
they find a good similarity between the matrices and show that the classical SPM allows them to predict the global
structure of the quantum SPM and to understand features of the latter in terms of classical aspects, i.e. by the analysis
of the trajectories that contribute to the corresponding features in the classical SPM.

On basis of these works and with our ability to compute the scattering matrix with a very large number of
modes ($\sim 1000$), as shown in Section 3, we propose to give in the present paper a similar study of the ray-wave
correspondence by comparing for two-dimensional curved waveguides the structures of the wave-scattering matrix
as defined in Section 3 and a ray-scattering matrix, counterpart of the wave-scattering matrix in the high frequency
asymptotic limit. The ray-scattering matrix is constructed in Section 4 and for several geometries of bends and
waveguides composed of bent and straight sections the wave- and ray-scattering matrices are given and their
structures compared.

2. Validation of the multimodal formulation at high frequencies: acoustic field

We discuss in the present section the suitability of the multimodal method proposed in Refs. [1,2] for the study
and computation of high frequency acoustic fields. Following this discussion, an example of calculation of a high
frequency acoustic field is given.

2.1. Direct calculation of the acoustic field

The basic idea in the formulation of the multimodal method is the possibility to use the local transverse
modes in a waveguide, whatever its overall geometry, as a basis on which the pressure and velocity can be
projected. Let $s$ be the coordinate along the axis of the waveguide, $\vec{w}$ the transverse coordinate, and $\psi_n(\vec{w}, s)$,
n $\in \mathbb{N}$, the local transverse modes, that is the eigenfunctions obeying the transverse eigenproblem $\Delta_\perp \psi_n = -\mu_n^2 \psi_n$
with the boundary condition $\psi_n(\vec{w}, s) = 0$ at the walls. The pressure and axial velocity are expressed using infinite
series

$$p = \sum_s \frac{1}{\sqrt{2\pi \rho}} p_s(s) \psi_s(\vec{w}, s), \quad v = \sum_s \frac{v_s(s)}{\sqrt{2\pi \rho}} \psi_s(\vec{w}, s).$$  \(1\)
\[ v_n = \sum \frac{1}{\sqrt{2 \pi}} U_0(s) \psi_n(s, x), \]

where \( \lambda_{ij}^2 = k^2 - \mu_{ij}^2 \), with \( k \) the wavenumber. Therefore, the equations of mass conservation and momentum conservation in the waveguide can be reformulated as functions of the vectors \( \vec{P} \) and \( \vec{U} \) of the components of the pressure and axial velocity in the basis (\( \psi_n s \in N \))[3]:

\[
\vec{P}' = A_{11} \vec{P} + A_{12} \vec{U},
\]

\[
\vec{U}' = A_{21} \vec{P} + A_{22} \vec{U},
\]

where \( \vec{P}' \) denotes the derivative with respect to \( s \) and \( A_{ij} \) are matrices depending on the frequency (\( k \)) and the geometrical parameters of the duct [1–3].

For a varying cross-section waveguide, and more generally when the properties of the waveguide (wall admittance, media fulfilling the duct, curvature, etc.) vary along the axis, the matrices \( A_{ij} \) are functions of \( s \) and Eqs. (3) and (4) must be integrated numerically. However, for reasons discussed by Pagneux et al. [3] and notably the numerical instability due the presence of evanescent modes, these equations cannot be integrated directly. Therefore an impedance matrix \( Z \) is defined, fulfilling \( \vec{P} = Z \vec{U} \). Substituting this relation into Eqs. (3) and (4), one shows easily that \( Z \) obeys the Riccati equation

\[
Z = A_{12} + A_{11} Z - Z A_{22} - Z A_{21} Z,
\]

which is numerically workable. Thus, the multimodal calculation of the acoustic field is subject to a prior calculation of the impedance matrix. The variations of \( Z \) or its inverse \( Y \) along the waveguide can present sharp peaks, making the use of an adaptive stepsize necessary for the integration of Eq. (5). Moreover, when the frequency, and consequently the number of modes, is increased, these peaks become more pronouncedly sharp for the components of \( Z \) corresponding to higher order modes. Thus the numerical integration is slowed down and becomes difficult to achieve.

However, for a waveguide with invariant properties along its axis, that is for a bend or a straight duct (we shall not detail this trivial last case), matrices \( A_{11} \) and \( A_{22} \) are null, \( A_{12} \) and \( A_{21} \) are constant. In the following of the paper two-dimensional bends will be considered; expressions of \( A_{12} \) and \( A_{21} \) in this case are given in Appendix A. Therefore a direct calculation of the acoustic field can be achieved, without prior numerical integration of the impedance matrix, allowing thus a great number of modes to be taken into account. Let first notice that the pressure \( P \) in such waveguide obeys the equation \( P'' + A_{12} A_{21} P = 0 \), of which a solution can be constructed in terms of the eigenvalues \( \nu_n^2 \) (\( n \in N \)) and eigenvectors \( \chi_n \) of the matrix \( A_{12} A_{21} \):

\[
P = XD(\nu_n^2) + XD^{-1}(\nu_n^2),
\]

where \( X = [\chi_0, \chi_1, \ldots] \), \( D(\nu_n^2) \) is diagonal and given by \( D_\nu(s) = \exp(-j \nu_n s) \), with \( \nu_n = \sqrt{\nu_n^2} \) if \( \nu_n^2 > 0 \) and \( \nu_n = -j \sqrt{-\nu_n^2} \) if \( \nu_n^2 < 0 \). \( \chi_1 \) and \( \chi_2 \) are constant column vectors, functions of the conditions at each end of the bend. Consider now a bend with a given radiation admittance \( Y_0 \) at the outlet (\( s = s_i \)) and a given pressure \( P_0 \) at the inlet (\( s = 0 \)). The continuity conditions at these two interfaces give the following system of four equations:

\[
P_0 = X \hat{C}_1 + X \hat{C}_2, \tag{7}
\]

\[
Y_0 \hat{P}_0 = -HY(\hat{C}_1 - \hat{C}_2), \tag{8}
\]

\[
\hat{P}_1 = X(\hat{D}_1 \hat{C}_1 + \hat{D}_1 \hat{C}_2), \tag{9}
\]

\[
Y_1 \hat{P}_1 = -HY(\hat{D}_1 \hat{C}_1 + \hat{D}_1 \hat{C}_2), \tag{10}
\]

whose unknowns are the constant vectors \( \hat{C}_1 \) and \( \hat{C}_2 \), the admittance \( Y_0 \) at the inlet of the bend, and the pressure \( \hat{P}_1 \) at the outlet. \( D = D(s), H \) and \( Y \) are functions of \( A_{12}, A_{21} \) and \( X \).
In the case of a bend ended by a semi-infinite straight duct, the admittance \( Y \) is then the characteristic admittance \( Y_c \), given by \( Y_{cmn} = (k_n/k)\delta_{mn} \). The determination of \( P_f \) and \( Y_0 \) allows us to define and calculate the reflection and transmission matrices of the bend, as shown in Ref. [1].

With the solutions \( \vec{C}_1 \) and \( \vec{C}_2 \) one determines the pressure (6) in the bend. However, as discussed in Ref. [1], the matrix \( D^{-1}(s) \) in (6) is a source of numerical problems of convergence. Thus, defining \( \tilde{\vec{C}}_2 = D^{-1}\vec{C}_2 \), the pressure in the bend is written as

\[
\vec{P} = X(D(s)\vec{C}_1 + D(s_f - s)\tilde{\vec{C}}_2),
\]

(11)

Fig. 1. Magnitude of the modal Gaussian beam in a two-dimensional bend connected to semi-infinite straight ducts. The harmonic point source \( \bullet \), of frequency such that 500 modes are propagative in the straight sections, is placed on one of the walls upstream from the bend. The excited modes are weighted by a Gaussian window, as plotted in Fig. 2, so that a Gaussian beam is emitted making an angle \( \pi/6 \) with the axis. The corresponding trajectory in the ray asymptotic limit is also plotted (---).
and this formulation depends on $D$ only, with positive arguments $s$ and $s_f - s$, and does not depend on $D^{-1}$. The solutions $\vec{C}_1$ and $\vec{C}_2$ are

\begin{align}
\vec{C}_1 &= (1 - \delta)^{-1} X^{-1} \vec{P}_0, \\
\vec{C}_2 &= -(Y_f X - HY)(Y_f X + HY)(1 - \delta)^{-1} X^{-1} \vec{P}_0,
\end{align}

where $\delta = D(Y_f X - HY)^{-1}(Y_f X + HY)D$.

2.2. Ray synthesis by a cluster of modes

The propagation of a cluster of modes in a rectilinear waveguide is known to show a ray-like behavior: modes interfere to produce a maximum of energy along two trajectories emanating from the source and following the path of the “modal rays” for the central mode in the group [6,7]. For a cluster emitted in a two-dimensional waveguide of width $h$ with a central mode $\psi_n$, these modal rays are the rays at angles $\theta_n = \sin^{-1}(n\pi/kh)$ and $-\theta_n$, classically used to describe the propagation of the mode $\psi_n$. In the following, we propose to study the propagation of such a beam emitted by a point source in a bent duct, in the high frequency regime ($\sim 500$ propagative modes).

Consider a two-dimensional bend of width $h$ joining two semi-infinite straight ducts and an harmonic point source, located upstream from the bend with coordinates $(r_s, s_s)$ in $(\vec{u}_r, \vec{u}_s)$ (Fig. 1). The incident wave $\vec{P}_i$ in the straight duct upstream from the bend is given by

\[ P_i(n) = \frac{j}{\sqrt{2\pi}} \psi_n(r_s) e^{-j kn(s - s_s)} e^{-(n_0 - n)^2/2\sigma^2} \forall n \in \mathbb{N}, \]

where $kn^2 = k^2 - (n\pi/h)^2$. In order to build a cluster of propagative modes, the components of $\vec{P}_i$ are weighted by a Gaussian window $\exp(-(n - n_0)^2/2\sigma^2)$, where $n_0$ is the index of the central mode in the mode group (Fig. 2).

Besides, the source in duct upstream is placed on one of the walls, so that a single beam is emitted (two beams, with directions $\theta_{n_0}$ and $-\theta_{n_0}$, are, a priori, emitted).

Fig. 1 shows the pressure field obtained with the direct calculation detailed above, for a bend with dimensions $R_0/h = 0.75$ and overall angle $\theta_f/R_0 = \pi/2$. Five hundred and fifty modes are taken into account in the numerical calculation, 500 of which are propagative. The Gaussian window, with a beam width parameter $\sigma = 31.6$, is centered on the mode $n_0 = 250$. With these high frequency and large spectral window, the beam obtained is narrow, its dispersion is weak, and the result approaches the asymptotic limit of rays: a ray of finite width has been synthesized and propagates in the bend according to the laws of geometrical acoustics. For comparison, a ray is plotted on the figure, emitted from the source point with a direction $\theta_{250} = \sin^{-1}(250\pi/kh) \sim \pi/6$ corresponding to the mode $n_0 = 250$: both ray and Gaussian beam follow the same trajectory.

![Plot of the spectral envelope of the incident Gaussian beam.](image-url)
Furthermore, one can notice the effect of the concave wall, which re-focus the beam, so that its width when reflecting the second time is similar to its original width. We guess that it could be possible, by a succession of reflections on concave walls, that the beam preserve its coherence and propagate indefinitely, without dispersion.

3. Validation of the multimodal formulation at high frequencies: scattering matrix

The present section is devoted to a validation at high frequencies of the algebraic formulation of the scattering matrix of a bend proposed in Refs. [1,2]. Two studies are proposed to validate this formulation. First the properties of unitarity and symmetry of the scattering matrix are checked when a large number of propagative modes is taken into account. Then the convergence of the proposed calculation is evaluated when the total number of modes taken into account is increased.

3.1. Properties of the scattering matrix

Consider a two-dimensional waveguide of arbitrary shape, connected to two straight ducts. The solutions of the wave equation \((\Delta + k^2) \psi = 0\) in these “left” (L) and “right” (R) ducts are

\[
\psi^{L,R}_n = \sum_{n \in \mathbb{N}} \frac{1}{\sqrt{2}} (A^{L,R}_n e^{-j k^{L,R}_n} + B^{L,R}_n e^{j k^{L,R}_n}) \psi^{L,R}_n(r),
\]  

where the functions \(\psi^{L,R}_n\) are the local transverse modes:

\[
\psi^{L,R}_n(r) = \sqrt{2 - \delta_{n,0}} \cos \left( \frac{n\pi r - h^{L,R}_n/2}{h^{L,R}_n} \right),
\]

with \(h^{L,R}\) the width of the “left” and “right” ducts.

The scattering matrix \(S\) relates outgoing waves to incoming waves:

\[
\begin{pmatrix} \vec{B}^L \\ \vec{A}^R \end{pmatrix} = S \begin{pmatrix} \vec{A}^L \\ \vec{B}^R \end{pmatrix},
\]

where \(\vec{A}^{L,R} = (A^{L,R}_n)_{n \in \mathbb{N}}\) and \(\vec{B}^{L,R} = (B^{L,R}_n)_{n \in \mathbb{N}}\) are column vectors. Thus, the \(S\) matrix can be written

\[
S = \begin{pmatrix} R & T \\ T & R \end{pmatrix},
\]

where \(R\) and \(T\) are the reflection and transmission matrices for a right-going incident wave \((\vec{A}^L)\) and \(R'\) and \(T'\) the reflection and transmission matrices for a left-going incident wave \((\vec{B}^R)\).

Time-reversal invariance of the operator \(\Delta + k^2\) and energy conservation require the \(S\) matrix to satisfy the following properties [8]:

\[
S = S, \quad S^\dagger M_0 S - S^\dagger M_1 + M_1 S = M_0,
\]

where ‘\(^\dagger\)’ and ‘\(^\prime\)’ denote the transpose and adjoint operators, respectively, and

\[
M_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & I - T \\ I - T & 0 \end{pmatrix}.
\]
where $I$ is the identity matrix and $I$ is diagonal and given by $I_n = 1$ for $0 \leq n < N_p$ and $I_n = 0$ for $n \geq N_p$, $N_p$ being the number of propagative modes. If one consider the propagative part of $S$ only, i.e. the matrices $R, T, R'$ and $T'$ restricted to the propagative modes at a given frequency – $S$ is then a $2N_p \times 2N_p$ matrix – Eq. (20) becomes

$$S^\dagger S = SS^\dagger = I,$$

(22)

$S$ is a unitary matrix.

The waveguides that will be considered in the sequel are composed of bent and straight sections. The algebraic calculation of the $S$ matrix of such waveguides has been given in Ref. [2]. In order to validate this calculation, in particular at high frequencies, when hundreds of modes are propagative in the straight sections, the properties (19) and (20) are evaluated by calculating the following quantities:

$$\epsilon_1 = \| S - S \| \| S \| ,$$

(23)

$$\epsilon_2 = \| S^\dagger M_0 S - S^\dagger M_1 S + M_1 S - M_0 \| \| M_0 \| ,$$

(24)

where $\| M \|$ is the Frobenius norm $\sqrt{\text{Tr}(M^\dagger M)}$, $\text{Tr}$ denoting the trace. For all cases considered in our study – 100, 500 and 1000 propagative modes, with the total number of modes (propagative and evanescent) taken into account varying between 100 and 1050 – errors $\epsilon_1$ and $\epsilon_2$ are in the order of magnitude of $10^{-13}$. These errors certainly depend on the precision of the computer only and are due to the “elementary” operations – product and inversion of matrices – carried out to calculate $S$. The symmetry and unitarity, in its generalized form (20), of the $S$ matrix are intrinsic properties of this matrix as it is formulated. Thus these are satisfied by construction.

### 3.2. Convergence of the multimodal formulation

A second validation of the calculation of the $S$ matrix at high frequencies is possible by checking the convergence of this calculation when the number $N$ of modes taken into account in the numerical calculation is increased. One consider a simple bend, which dimensions are $R_0/h = 0.75$ and $s/R_0 = \pi/2$, and the rate of convergence is

![Fig. 3. Convergence of the algebraic calculation of the scattering matrix. (×) $N_p = 100$, (◦) $N_p = 500$, (□) $N_p = 1000$.](image-url)
evaluated by measuring the error
\[ \epsilon = \frac{\| S_N - S_{N_{1050}} \|}{\| S_{N_{1050}} \|} \] (25)
as a function of \( N \), for a fixed value of the number \( N_p \) of propagative modes. The notation \( S_N \) denotes the propagative part of the \( S \) matrix calculated with \( N \) modes at a frequency such that \( N_p \) modes are propagative. The matrix \( S_{N_{1050}} \) is chosen as reference.

Calculated for three different frequencies (corresponding to \( N_p = 100, 500 \) and 1000), the error \( \epsilon \) decreases when increasing the number \( N - N_p \) of evanescent modes taken into account, following a \( 1/(N - N_p)^2 \) pattern (Fig. 3). The convergence is thus fast, and a small number of evanescent modes in the numerical calculation is sufficient to ensure the accuracy of the result.

4. Ray-wave correspondence of the scattering matrix

In the present section, the ray-wave correspondence of the scattering matrix (\( S \) matrix) is studied by first constructing a ray-\( S \) matrix and then comparing its structure with the structure of the wave-\( S \) matrix that is obtained with the exact algebraic formulation (see Section 3 and Refs. [1,2], for different geometries of curved waveguides.

In this section, the notation \( S \) will be used to denote the scattering matrix restricted to its propagative part.

4.1. Building a ray-\( S \) matrix

Since each mode \( \psi_n \) in a rectilinear waveguide of width \( h \) is characterized by a propagation constant \( k_n^2 = k^2 - (n\pi/h)^2 \), we can associate – so long as the mode is propagative – the angle \( \theta_n \) between the wavenumber \( \vec{k} \) and its longitudinal component \( k_n^2 \):
\[ \theta_n = \sin^{-1} \left( \frac{n\pi}{kh} \right) \] (26)
Therefore, following Luna-Acosta et al. [4] and Méndez-Bermúdez et al. [5] the ray-\( S \) matrix of any waveguide section can be built as follows: consider a particle entering from the left in this waveguide, making an angle \( \theta_i \) with the waveguide axis. Since the number of propagative modes – the frequency – is finite, there would not be any mode, in almost any case, that will correspond to this particular angle. However, \( \theta_i \) being within a range \([\theta_n, \theta_{n+1}]\), \( n \in \mathbb{N} \), we associate the mode \( n \). Then the particle propagates in the waveguide, describing a ray that is reflected on the walls according to laws of geometrical acoustics, before exiting to the left or to the right, making an angle \( \theta_e \) within a range \([\theta_m, \theta_{m+1}]\), \( m \in \mathbb{N} \). To this angle we associate the outgoing mode \( m \). By varying the angle \( \theta_i \) and the initial position of the particle so that to account for a great number of trajectories (rays), we obtain for each value of \( n \) a distribution of values of \( m \). We also distinguish the rays exiting to the left from those exiting to the right to give the contribution of the incident mode \( n \) to the reflected or transmitted mode \( m \), that is \( |R_{mn}|^2 \) and \( |T_{mn}|^2 \). Considering a particle entering from the right in the waveguide, the same process leads to \( |R'_{mn}|^2 \) and \( |T'_{mn}|^2 \).

4.2. Waveguide model

Our aim is to compare at high frequency the ray- and wave- structures of the \( S \) matrix of curved waveguides, composed of bent and straight sections (see, e.g., Fig. 4). The waveguide is connected to straight ducts at each extremity, and rays are emitted from one or the other of these ducts, with the following initial conditions: the position \( y_i \in [0, 1] \) of the particle on the section of the waveguide and the angle \( \theta \in [-\pi/2, \pi/2] \). One can already notice a first characteristic of such waveguides: since the cross-section is constant, a ray that enter in the waveguide
Fig. 4. Waveguide model.

Fig. 5. Traces of $R^\dagger R$ and $T^\dagger T$ as functions of the number of propagative modes $N_p$, for a bend of dimensions $R_0/k = 0.75$ and $a_0/R_0 = \pi/2$.

Fig. 6. Geometries of the waveguides.
cannot propagate back to the duct from which it was emitted. As a consequence, blocks \( R \) and \( R' \) of the ray-S matrix are null. Below we see that this result is coherent with the properties of the wave-S matrix. Indeed, it has been shown that the energy conservation implies the relation \( S^\dagger S = SS^\dagger = I \); for a bend – one have then \( R = R' \) and \( T = T' \) because of the symmetry of the bend – this relation implies \( R^\dagger R + T^\dagger T = I \). Thus,

\[
\text{Tr}(R^\dagger R) + \text{Tr}(T^\dagger T) = N_p. \tag{27}
\]

Fig. 5 shows traces \( \text{Tr}(R^\dagger R) \) and \( \text{Tr}(T^\dagger T) \) as functions of \( N_p \). The terms of \( R \) becomes rapidly small, as soon as the frequency is high enough to allow several modes to propagate. Thus, as the geometrical approach shows it,
the reflection of a wave in a bend, and consequently in any duct system composed of bent and straight sections, is negligible at high frequency.

4.3. Results

In the present section, four geometries of waveguides are considered (Fig. 6): two simple bends, one of which is weakly curved and the other more sharply curved, and two waveguides composed of four bends joined with straight sections. Again, one of these duct system is weakly curved while the other is more sharply curved. In each case, we give the ray-T matrix, obtained by the procedure described above with a set of $10^6$ or $10^7$ pairs of initial conditions.

Fig. 8. (a) Ray- and (b) wave-T matrix of a bend of dimensions $R_0/h = 0.75$ and $\alpha_i / R_0 = \pi/2$. One thousand and fifty modes are taken into account in the calculation of $O_{\alpha i}$, 1000 of which are propagative.
(\(\psi_i, \theta_i\)), and the wave-T matrix obtained with the multimodal formulation. Reflection matrices are not given, since it has been shown that the reflection in such waveguides at high frequency is negligible.

4.3.1. Weakly curved bend

The waveguide that is studied is a bend which dimensions are \(R_0/h = 4\) and \(s_f/R_0 = \pi/2\) (Fig. 6(a)), and the frequency is such that 1000 modes are propagative. One thousand and fifty modes are taken into account in the multimodal calculation of the wave-T matrix (Fig. 7).

At such frequency, the similarity between the ray- and wave-matrices is remarkable. Nevertheless, a good similarity can also be observed for a lower number of propagative modes, for a few hundreds but also for about ten modes. The studies by Luna-Acosta et al. on open chaotic billiards have notably been achieved for 30 and 200 modes. The ray-approach, whose principle is very simple – only geometrical aspects are considered to propagate the rays in the waveguide and neither the phase information nor diffraction or interference effects are taken into account – gives a matrix showing all the complex patterns that can be observed on the wave-transmission matrix. From this ray-matrix, it is then possible to predict important features of the wave transmission by such waveguides. Note however that the ripples on the wave-matrix structure, due to the interference effects, do naturally not appear in the ray-matrix.

The global structure of the matrix, localized in the upper right part near the first diagonal, shows that the rays entering with a large angle of incidence \(\theta_i\) transmit without significant deviation. Corresponding incident modes, of high orders, will thus transmit predominantly onto the same mode. The propagation of higher order modes is weakly influenced by the curvature of the waveguide, contrary to lower order modes, which transmit significantly onto modes off the diagonal, as the ray-T matrix also predicts it.

4.3.2. Sharp bend

For a large curvature of the bend (Fig. 6(b)), the structure of the transmission matrix is strongly modified (Fig. 8). The dispersion of the ray trajectories is important, giving the matrix its spread structure.

Another modification of the matrix, and consequently of the scattering properties, is the emergence of a high-intensity arc whose center of curvature is the point \((n, m) = (0, 0)\). This high-intensity arc is due to trajectories
Fig. 10. Ray-T matrix of a waveguide composed of 1, 2, 3, and 4 bends of dimensions $R_0/h = 4$ and $s_f/R_0 = \pi/2$, connected with straight sections of length $h$. 
that transmit without colliding the curved walls of the bend. The mode $n$ thus transmit predominantly onto the mode $N_p - 1 - n$, except for the few higher order modes (see the top right corner of the matrix) which transmit predominantly onto the same mode, as in the weakly curved bend.

Besides these trajectories that transmit directly without collision on the walls, another type of trajectories gives the ray-$T$ matrix an interesting structure: trajectories that collide only on the concave wall of the bend, following the so-called whispering gallery orbits. It has been shown (see Refs. [5, 9]) that such orbits produces a self-similar structure, notably in the $S$-matrix. The enlargement of the bottom left part of the ray-$T$ matrix in Fig. 9 shows such self-similar structure. The point of convergence of this structure is found for $\theta_i = 0$, the angle at which a

![Fig. 11. (a) Ray- and (b) wave-$T$ matrix of a waveguide composed of 4 bends of dimensions $R_0/h = 4$ and $\alpha/R_0 = \pi/2$, connected with straight sections of length $h$. Five hundred and fifty modes are taken into account in the calculation of (b), 500 of which are propagative.](image-url)
ray, colliding the concave wall at the entrance of the bend, will collide an infinite number of times before being transmitted at an angle $\theta_e = 0$. Thus, one can confirm the existence of whispering gallery modes in the waveguide.

4.3.3. Four weakly curved bends

When the waveguide is composed of a succession of several bends connected by straight ducts, instead of a single bend, the ray dynamics is modified. Fig. 10 shows the ray-$T$ matrix for a waveguide composed of 1, 2, 3 then 4 bends, each with the same dimensions: $R_0/h = 4$, $s_f/R_0 = \pi/2$. Apart from the top right corner, corresponding

Fig. 12. (a) Ray- and (b) wave-$T$ matrix of a waveguide composed of 4 bends of dimensions $R_0/h = 0.75$ and $s_f/R_0 = \pi/2$, connected with straight sections of length $h$. Five hundred and fifty modes are taken into account in the calculation of (b), 500 of which are propagative.
to the higher order modes, the structure of the matrix spread out on both sides of the first diagonal when the number of bends is increased. Besides, the very clear structures that can be distinguished on the 1-bend matrix are gradually substituted to a large quasi-homogeneous area. A homogeneous area results when rays entering the waveguide within a range $\Delta \theta_{1}$ transmit uniformly throughout a much wider range $\Delta \theta_{e}$, denoting a strong sensitive dependence to initials conditions. The ray dynamics, regular in a single bend, becomes, or seems to become irregular when the waveguide is composed of a succession of bent and straight sections. As for many quantum billiards (Bunimovich stadium, Sinai billiard, cut-circle billiard or rippled billiard as in Refs. [4,5]), these are the joint effects of straight and curved walls which leads to an irregular ray dynamics.

However, a precise characterization of the ray dynamics – regular, mixed or fully chaotic – has still to be done and the simple observation of the scattering matrix allows us only to suspect the emergence of an irregular dynamics. Moreover, patterns are still discernible on either the ray- and wave-matrices of the 4-bends system (Fig. 11).

4.3.4. Four sharp bends

The ray- and wave- $T$ matrices shown in Fig. 12 are those of a 4-bends system similar to the previous one, but with a larger curvature of the bends ($R_0/\varrho = 0.75$, see Fig. 6(d)). The ray dispersion is important: whatever the incident angle $\theta_{i}$, rays transmit uniformly throughout the whole range $[-\pi/2, \pi/2]$. Again, one can suppose that the ray dynamics is irregular in this waveguide system.

Some contrasted patterns, however, are discernible, among these are three small segments, noted (1), (2) and (3) in Fig. 12(a), on the first diagonal. These particular patterns are formed by trajectories that transmit without colliding any curved wall of the waveguide (Fig. 13). These orbits are very stable, i.e. have a low sensitive dependence on initial conditions: small variations of these conditions will lead to closely the same trajectory and a very similar angle $\theta_{e}$ at the exit.

5. Conclusion

The suitability of the multimodal formulation of the wave propagation in bends for investigating very high frequency problems has been shown. Since the multimodal equations in a circular bend of constant cross-section are amenable to an algebraic solutions, for both the acoustic field and the scattering matrix, and since these algebraic solutions involve only simples operations on matrices, a large number of modes can be taken into account, and therefore the computation at high frequencies of acoustic fields or scattering matrices is possible. Moreover, as neither the algorithms nor the computer capacities have been optimized for this computation with large matrices, calculations with a larger number of modes (~2000) can be reasonably planned.

Following this validation, a study of the ray-wave correspondence of the scattering matrix for two-dimensional curved waveguides composed of bent and straight sections has been presented. Structures of the ray- and wave-
scattering matrices have been qualitatively compared, showing a remarkable similarity between these two matrices. Thus the scattering matrix constructed in the ray asymptotic limit allows us to predict and understand numbers of the wave scattering properties in curved waveguides. One can expect to increase the similarity between the ray- and wave-matrices and to allow a quantitative comparison by taking into account the phase information in the ray method.

Appendix A

Consider a bend of width \( h \) and axis curvature \( \kappa = 1/R_0 \) as shown in Fig. 1. The transverse eigenmodes are
\[
\psi_n(r) = \sqrt{2} - \delta_n \cos(n\pi(r - h/2)/h)
\]
and \( A_{12} \) and \( A_{21} \) are given by
\[
A_{12} = -jEBE^{-1},
\]
\[
A_{21} = \frac{1}{j}E(C + KB)E^{-1},
\]
where matrices \( E \) and \( K \) are diagonal and given by
\[
E_n = \sqrt{jkn}
\]
and
\[
K_n = k_n^2,
\]
with \( k_n^2 = k_n^2 - (n\pi/h)^2 \).

B and \( C \) are given by
\[
B_{mn} = \begin{cases}
1 & \text{if } m = n, \\
\sqrt{2} - \delta_m \sqrt{2} - \delta_n \frac{h}{\pi} (-1)^{m+n} - 1 & \frac{m^2 + n^2}{(m^2 - n^2)^2} \text{ if } m \neq n,
\end{cases}
\]
\[
C_{mn} = \begin{cases}
0 & \text{if } m = n, \\
\sqrt{2} - \delta_m \sqrt{2} - \delta_n \frac{h}{\pi} (-1)^{m+n} - 1 & \frac{m^2 + n^2}{m^2 - n^2} \text{ if } m \neq n,
\end{cases}
\]

References