Revisiting the edge resonance for Lamb waves in a semi-infinite plate

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The resonance for the elastic plate with a free edge is studied from the point of view of complex resonance. The variations of the real part and of the imaginary part of the complex resonance frequency as a function of the Poisson ratio \( \nu \) are determined numerically. The results confirm the real resonance frequency theoretically predicted in I. Roitberg et al., Q. J. Mech. Appl. Math. 51, 1–13 (1998) for a zero Poisson ratio \( \nu_1 = 0 \), and a real resonance frequency that corresponds to a Lamé mode is discovered for a Poisson ratio \( \nu_2 = 0.2248 \). It is shown that both real resonance frequencies may exist, at these two particular values of \( \nu \), because of the decoupling between the propagating Lamb mode and the set of evanescent Lamb modes. © 2006 Acoustical Society of America. [DOI: 10.1121/1.2214153]

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I. INTRODUCTION

The study of Lamb wave propagation in elastic waveguides is an active domain of research, notably because of the important applications it may have in nondestructive testing. It remains also an interesting domain of research on its own because of its intrinsic and complicated vectorial character due to the coupling between longitudinal and transversal modes. It is remarkable that, for such a venerable subject, there remain fundamental open questions; e.g., the mathematical proof of the completeness of the Lamb modes has not yet been achieved entirely.

Another fundamental problem for Lamb waves that is not fully understood is the edge resonance for a semi-infinite plate with a free edge. This resonance occurs for symmetric vibration at frequencies such that only the lowest order symmetric Lamb mode \( S_0 \) can propagate, and it is characterized, near a particular frequency, by the abrupt change of the reflection coefficient of the mode \( S_0 \) and by the excitation of a localized motion of the plate at the edge. Actually, in Refs. 3–9, it appears as a quasiresonance with a large but finite amplitude response when the incident wave is the \( S_0 \) mode, and for Poisson ratio near 0.3. That resonant phenomenon was discovered in an experiment by Glazis in 1956. Thereafter, several authors confirmed the existence of this resonance owing to numerical calculations based on modal or finite-element methods. Recently, new experimental and numerical studies have shown the same kind of resonances, and the results in Ref. 9 suggest that the resonance persists by coupling symmetric and antisymmetric Lamb modes when the free edge is beveled. Other geometries corresponding to rods or cylindrical shells have also been shown to display the same kind of resonant behavior. Mathematically, the edge resonance for semi-infinite plate has been studied by Roitberg et al. who were able to apply the tools of functional analysis to obtain a proof of the existence of a trapped mode for a real frequency in the particular case of zero Poisson ratio \( \nu = 0 \). One fundamental idea in their paper was to use the absence of coupling between the only propagating Lamb mode \( S_0 \) and the remaining infinity of evanescent Lamb modes at \( \nu = 0 \).

In this paper, the aim is to bridge the gap between the particular result of Ref. 12 corresponding to a real resonance frequency at \( \nu = 0 \) and the other results corresponding to a quasiresonance and obtained for much higher Poisson ratio \( \nu \sim 0.3 \). We use the concept of complex resonance that has been widely used in acoustics in the resonance scattering theory, and that corresponds to a resonance frequency for which a nonzero imaginary part is allowed. For the semi-infinite plate, a complex resonance is associated with a pole in the complex plane of the reflection coefficient of the incident mode \( S_0 \), and it corresponds also to an eigenmode with a complex frequency of the elasticity equations without incident wave. Since the problem, when cast in dimensionless form, has only two parameters, the dimensionless frequency and the Poisson ratio, we seek complex resonance frequencies as a function of \( \nu \) in the physically acceptable range of Poisson ratio \( \nu = 0 \) to \( \nu = 0.5 \).

The paper is organized as follows. The scattering problem is formulated in Sec. II. Section III presents the behavior of the real and imaginary parts of the complex resonance frequency as a function of the Poisson ratio, and the resonant mode is displayed. In Sec. IV, we discuss the results, and the concluding remarks are presented in section V. The numerical method used to obtain the results is briefly described in the Appendix.

II. FORMULATION OF THE PROBLEM

The problem under study corresponds to the bidimensional geometry shown in Fig. 1, with the vertical edge at \( x = 0 \) and the horizontal surfaces at \( y = \pm h \). Symmetric in-plane displacements are considered and the frequency is such...
that only the first Lamb mode $S_0$ is propagating. The harmonic time dependence $e^{i\omega t}$ will be omitted in the following. The elasticity equations are

$$\rho \omega^2 \mathbf{w} = \nabla \cdot \mathbf{\sigma},$$

where $\rho$ is the density, $\mathbf{w} = (u, v)^T$ is the displacement, and

$$\mathbf{\sigma} = \begin{pmatrix} s & t \\ t & r \end{pmatrix},$$

the stress tensor, with

$$s = \lambda \partial_x u + (\lambda + 2 \mu) \partial_y u, \quad t = \mu (\partial_x u + \partial_y v), \quad r = (\lambda + 2 \mu) \partial_y u + \lambda \partial_x u,$$

where $(\lambda, \mu)$ are the Lamé’s constants. The boundary conditions correspond to traction-free surface of the plate, $\mathbf{\sigma} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the normal to the surface boundary, and are given by $t = r = 0$ for $y = \pm h$, $x = 0$, $r = s = 0$ for $x = 0$.

These equations can be made dimensionless by renormalizing the components of the displacement $\mathbf{w}$ by $h$, the components of the stress tensor $\mathbf{\sigma}$ by $\mu$, and the coordinates $x$ and $y$ by $h$. The resulting dimensionless equations are

$$-\Omega^2 u = \partial_x s + \partial_y t, \quad -\Omega^2 v = \partial_x t + \partial_y r,$$

$$s = \gamma \partial_y u + (\gamma - 2) \partial_x u, \quad t = \partial_y v, \quad r = (\gamma - 2) \partial_x u + \gamma \partial_y v,$$

with dimensionless frequency $\Omega = \omega h / c_T$ and $\gamma = c_T^2 / c_L^2 = (\lambda + 2 \mu) / \mu \geq 2$, where $c_T = \sqrt{\mu / \rho}$ and $c_L = \sqrt{(\lambda + 2 \mu) / \rho}$ are, respectively, the transversal and longitudinal wave speeds. The boundary conditions are

$$r = t = 0 \quad \text{at} \quad y = \pm 1, \quad s = 0 \quad \text{at} \quad x = 0.$$

It is important to note that the set of equations (4)–(6) has only two parameters, which are the frequency $\Omega$ and the Poisson ratio $\nu = (\gamma - 2) / [2(\gamma - 1)]$ with $0 < \nu < 0.5$. In Eq. (5), \gamma is expressed as a function of $\nu$ by $\gamma = 2(1 - \nu) / (1 - 2\nu)$.

Since we are interested in the scattering of the propagating mode $S_0$ by the free edge, the solution can be written\(^16\) in the form of an incident left-going Lamb mode $S_0$ and a reflected right-going Lamb mode $S_0$ plus right-going evanescent Lamb modes $S_n (n \geq 1)$,

\[ \begin{bmatrix} u \\ v \\ s \\ t \end{bmatrix} = e^{-ik_0t} \begin{bmatrix} u_0^- \\ v_0^- \\ s_0^- \\ t_0^- \end{bmatrix} + Re^{i\omega t} \sum_{n=1}^{+\infty} a_n e^{i\omega t} \begin{bmatrix} u_n^+ \\ v_n^+ \\ s_n^+ \\ t_n^+ \end{bmatrix}. \]

The left-going Lamb $S_0$ mode corresponds to eigenvectors $[u_0^-(y), v_0^-(y), s_0^-(y), t_0^-(y)]^T$ with wave number $-k_0$, and the right-going Lamb modes correspond to eigenvector $[u_n^+(y), v_n^+(y), s_n^+(y), t_n^+(y)]^T$ with wave number $k_n (n \geq 0)$. To completely define $R$ we need to specify that we use the symmetry\(^17,16\) between right-going and left-going Lamb modes such that $(u_n^-, t_n^-)^T = (u_n^+, t_n^+)^T$ and $(s_n^-, v_n^-)^T = -(s_n^+, v_n^+)^T$.

To calculate the solution of the problem, we use a hybrid method with a collocation discretization along the coordinate $y$ and a modal approach along the coordinate $x$. This method allows us to find the reflection coefficient $R$ and the coefficients $a_n$. For the sake of clarity, it is described in the Appendix.

For real $\Omega$, the conservation of energy implies\(^17\) that $|R(\Omega, \nu)| = 1$. Figure 2 shows an example for $\nu = 0.3$ of the behavior of the real part of $R$ as a function of frequency. It is the classical type of result that has been shown in earlier paper on the edge resonance (see Refs. 4–8 for instance); it displays an abrupt change of $R$ in a narrow range of frequencies typical of the edge resonance.\(^5\) This is the typical behavior for a quasiresonance.

For complex $\Omega$, $R$ is not constrained to be on the unit circle, and complex resonances correspond to frequency where $R$ is infinite. These poles of $R$ in the complex $\Omega$ plane have negative imaginary part $\text{Im}(\Omega) < 0$ (see Ref. 15 for a clear discussion) and they correspond to a solution (7) which is an eigensolution of the elasticity equations with only right-going waves. This eigenmode has a temporal dependence $e^{-i\omega t}$ and is thus decaying with time at a rate which is given by the imaginary part of the complex resonance frequency. In physical units, the decaying rate (the inverse of the “ringing time”) is given by $\alpha = (c_T / h) \text{Im}(\Omega_R)$, where $\Omega_R$ is the com-
complex resonance frequency. Moreover, to poles $\Omega_R$ of $R$ with $\text{Im}(\Omega_R) < 0$ correspond the complex conjugates $\bar{\Omega}_R$, with $\text{Im}(\bar{\Omega}_R) > 0$, which are zeros of $R$ such that $R(\bar{\Omega}_R) = 0$. Numerically we seek these zeros of $R$ by a Newton-Raphson method.

III. RESULTS

In this section we present the results of the problem posed in Sec. II owing to the numerical method described in the Appendix. Numerically, one and only one complex resonance frequency $\Omega_R$ has been found for each value of the Poisson ratio $\nu$. Figures 3–5 display the principal results of that paper. They show the behavior of the complex resonance frequency as a function of the Poisson ratio. The real part of the complex resonance frequency (Fig. 3) varies almost linearly with $\nu$. For practical purpose, a very good approximation of this real part is given by the empirical formula

$$\text{Re}(\Omega_R) = 0.652\nu^2 + 0.898\nu + 1.9866,$$

whose error is less than $10^{-3}$. The values of the real resonance frequencies existing in the literature are recovered and are very well approximated by the formula (8). This correspondence between the real part of the complex resonance frequency computed in this work and the real frequency obtained in Refs. 4–9 by looking at real frequencies the local behavior of $\text{Im}(R)$ is guaranteed by the very small values of the imaginary part of the complex resonance frequency. This fact is confirmed in Fig. 4, where the imaginary part of $\Omega_R$ is plotted as a function of the Poisson ratio. It can be seen that $\text{Im}(\Omega_R)$ remains close to zero for $\nu < 0.25$ and decreases substantially for higher $\nu$. To gain insights into the behavior of $\text{Im}(\Omega_R)$ in the range $0 < \nu < 0.25$, Fig. 5 shows a magnified view of Fig. 4. It appears that $\text{Im}(\Omega_R)$ is zero for two values of the Poisson ratio: $\nu_1 = 0$ and $\nu_2 = 0.2248$. The first value, $\nu_1 = 0$, corresponds to the real frequency resonance discovered in Ref. 12. The second value, $\nu_2 = 0.2248$, corresponds to numerical accuracy to a resonance frequency $\Omega_R = \pi \sqrt{2}$, i.e., to an $S_0$ Lamé mode as will be seen in the following section. Note that if the value $\Omega_R = \pi \sqrt{2}$ is used in Eq. (8), it yields a second-degree equation with unknown $\nu$, the solution of which is $\nu_1 = \nu_2 = 0.2248$. Note also that near the two real resonance frequencies the local behavior of $\text{Im}(\Omega_R)$ is $\text{Im}(\Omega_R) \sim C_1 \nu^2$ when $\nu \to \nu_1 = 0$ and $\text{Im}(\Omega_R) \sim C_2 (\nu - \nu_2)^2$ when $\nu \to \nu_2$, where $C_1$ and $C_2$ are constants. It would be interesting to theoretically find these asymptotic expressions with the techniques presented in Ref. 15. In the same manner as for the real part of $\Omega_R$, the imaginary part is well approximated by the empirical formula

$$\text{Im}(\Omega_R) = -\frac{C \nu^2 (\nu - \nu_2)^2}{1 + \left(\frac{\nu - \nu_3}{a}\right)^2 + \left(\frac{\nu - \nu_3}{b}\right)^2},$$

where $\nu_3 = 0.2062$, $a = 0.1696$, $b = 0.2606$, and $c = 1/0.0313$. This empirical formula has been guessed owing to the
known local behavior of $\text{Im}(\Omega_R)$ near $\nu_1$ and $\nu_2$ and gives an error less than 1% on the full range of Poisson ratio. Actually, if the curves associated with Eqs. (8) and (9) were plotted on the corresponding figures, it could not be distinguished from the curves already displayed. A quite remarkable point is the symmetry with respect to $\nu_1$ in the expression (9), and it remains an open question to know if it occurs only by chance or if it is due to some hidden mathematical symmetry in the problem.

Another quantity of interest is the background phase $\phi$ that can be defined for real frequency $\Omega$ by the relation

$$R=\frac{\Omega-\bar{\Omega}_R}{\Omega-\Omega_R}e^{i\phi}.$$  \hspace{1cm} (10)

This relation is valid for real frequency $\Omega$, i.e., when $|R|=1$, and it reflects the fact that the complex $\Omega_R$ and its conjugate $\bar{\Omega}_R$ are, respectively, a pole and a zero of $R$. The background phase $\phi$ represents the slow variation of the reflection coefficient superposed on the rapid variation due to the presence of the pole $\Omega_R$. Figure 6 shows the behavior of $\phi$ on the whole range of Poisson ratio $\nu$ and for real frequency $\Omega$ sustaining one propagating Lamb mode (note that in the limit $\nu\rightarrow 0.5$ the first cut-on frequency for higher order modes is known\(^6\) to be $\nu_c=\pi$). It can be seen that the values of the background phase $\phi$ are very low in the major part of the $(\nu,\Omega)$ plane, with a maximum amplitude of 0.3 for the higher Poisson ratio (around 0.45). Consequently, because of the low value of $\phi$, the variations of $R$ are almost entirely due to the presence of the complex resonance frequency $\Omega_R$.

Figures 7–10 represent the patterns of the elastic fields $u(x,y)$, $v(x,y)$, $s(x,y)$, and $t(x,y)$ at the second real resonance frequency $\Omega_R=\pi/\sqrt{2}$ with $\nu=\nu_2=0.2248$. These patterns are almost the same as for the other real resonance frequency $\Omega_R=1.9866$ with $\nu=\nu_1=0$. Because of the symmetry of the motion, only the upper part, $y>0$, of the geometry, $x\geq 0$ and $|y| \leq 1$, of the semi-infinite plate is shown. In Figs. 7 and 8, it can be seen that the motion is localized near the edge and the largest amplitudes are at the corners. Incidentally, the representation in Figs. 9 and 10 of the two components, $s$ and $t$, of the stress tensor permits us to verify that they satisfy the boundary condition, $s=t=0$ at $x=0$ and $t=0$ at $y=1$. It is interesting to note that the lines of equal values of $t$ form small closed contours in the vicinity of the corner; it has to be associated with the type of local behavior of the stress at corner (stress singularity) which is similar to the corner eddies for creeping flows in fluid mechanics\(^19\). An illustration of the shape of the resonant mode for the entire plate is given in Fig. 11 and can help one have a clearer view of the phenomenon. In this figure, the amplitude of the mode has been chosen arbitrarily so that the shape of the motion can be seen easily.

**IV. INTERPRETATION OF THE RESULTS FOR THE TWO REAL RESONANCES**

In the preceding section it has been shown that there exists a complex resonance frequency for $0\leq\nu<0.5$ and,
that for two particular values of the Poisson ratio, \( \nu_1 = 0 \) and \( \nu_2 = 0.2248 \), the imaginary part of the complex resonance frequency is zero. In the following, the existence of the two real resonance frequencies for \( \nu_1 \) and \( \nu_2 \) is explained by showing that in both cases there is a decoupling between the propagating Lamb mode \( S_0 \) and the higher order evanescent Lamb modes.

The solution in term of Lamb modes [Eq. (7)] can be rewritten as

\[
\begin{align*}
X(x, y) &= \left( X_0(y) \right) e^{-ik_0x} + \sum_{n=0} a_n \left( X_n(y) \right) e^{ik_nx}, \\
Y(x, y) &= \left( -Y_0(y) \right) e^{-ik_0x} + \sum_{n=0} a_n \left( -Y_n(y) \right) e^{ik_nx},
\end{align*}
\]

with \( a_0 = R \) and where \( X = (u, t)^T, Y = (-s, v)^T \) and \( X_n = (u_n^+, t_n^+)^T, Y_n = (-s_n^+, v_n^+)^T \). The symmetry between right-going and left-going modes has been used so that \( (u_n^+, t_n^+)^T = X_n \) and \( (-s_n^+, v_n^+)^T = -Y_n \). The interest of this formulation is that it can be easily projected owing to the biorthogonality relation\(^7,16,20\)

\[
\int_{-h}^h X_n \cdot Y_m dy = J_n \delta_{nm}.
\]

To find the unknown components, \( a_n \) \((n \geq 0)\), the two lines of Eq. (11) are projected\(^16\), using Eq. (12), at \( x = 0 \) where \( t = 0 \) and \( s = 0 \). The following system of equations is then obtained:

\[
\begin{align*}
J_m(\delta_{0m} + a_m) &= -\sum_{n=0} (a_n + \delta_{nm}) \int_{-h}^h u_n^+ s_m^+ dy, \\
J_m(-\delta_{0m} + a_m) &= \sum_{n=0} (a_n - \delta_{nm}) \int_{-h}^h v_n^+ t_m^+ dy.
\end{align*}
\]

Generally, this system of equations is not easy to solve because when truncated to a finite size, say \( N \), it seems to be redundant with \( 2N \) equations and \( N \) unknowns \( a_n \). Actually, this intricacy comes from the boundary conditions at \( x = 0 \) which impose one component of \( X \) \((t = 0)\) and one compo-
ment of \( \mathbf{Y} \) \((s=0)\). If the boundary conditions were entirely on \( \mathbf{X} \) or on \( \mathbf{Y} \) (often called mixed boundary conditions; see for instance Ref. 21), the resulting system would not be redundant and the situation would be easier. The difficulty to solve the system (13) and (14) is one reason why, in this particular free-edge problem, we choose another numerical method (see the Appendix) to solve the scattering problem. Nevertheless, the system is greatly simplified for \( \nu=0 \) and for Lamé mode where \( \Omega=\pi/\sqrt{2} \) and \( \nu \neq 0 \). In fact, these two cases correspond to the two real resonance frequencies found in Sec. III.

For the first case where \( \nu=0 \), studied in Ref. 12, Eqs. (4) and (5) are greatly simplified since \( \gamma=2 \). It appears that the mode \( S_0 \) is such that \( u_0^+=1, r_0^+=0, s_0^+=2ik_0y, \) \( \nu_0^+=0 \), and \( k_0=\Omega/\sqrt{2} \). For the second case, where \( \Omega=\pi/\sqrt{2} \) and \( \nu \neq 0 \), the mode \( S_0 \) is a Lamé mode\(^21\) such that \( u_0^+=d\phi_0/dy, r_0^+=0, s_0^+=2ik_0y, \) \( \nu_0^+=-ik_0\phi_0, \) and \( k_0=\Omega/\sqrt{2}=\pi/2 \), where \( \phi_0(y) \) is the transverse potential. The Lamé mode is a purely transversal wave that corresponds to transversal plane waves reflecting on the horizontal boundaries with angle \( \pi/4 \) and for which there is no mode conversion with longitudinal waves. In fact, the first case is also a Lamé mode with \( k_0=\Omega/\sqrt{2} \), but it is a degenerate Lamé mode for the special case of zero Poisson ratio \( \nu \); it is purely longitudinal with a constant longitudinal potential \( \phi_0(y)=C \).

For both cases, by using the biorthogonality relation (12) and the fact that \( \nu_0^+=0 \), it can be verified that we have the following useful relations:

\[
\int_{-h}^{h} u_0^+ s_0^- dy = \int_{-h}^{h} u_0^+ s_0^+ dy = -J_0 \phi_0, \tag{15}
\]

\[
\int_{-h}^{h} v_0^+ r_0^- dy = \int_{-h}^{h} v_0^+ r_0^+ dy = 0. \tag{16}
\]

Then, Eq. (13) gives no information for \( m=0 \) and becomes, for \( m \geq 1 \),

\[
J_m a_m = -\sum_{n=1}^{m} a_n \int_{-h}^{h} u_n^+ s_n^- dy, \tag{17}
\]

which involves only the evanescent modes and is thus decoupled from the propagating mode \( S_0 \). Equation (14) for \( m \geq 1 \) becomes

\[
J_m a_m = \sum_{n=1}^{m} a_n \int_{-h}^{h} v_n^+ r_n^- dy. \tag{18}
\]

For \( m=0 \), i.e., the projection on mode \( S_0 \), Eq. (14) becomes

\[
J_0 (a_0 - 1) = 0, \tag{19}
\]

which yields immediately \( R=a_0=1 \).

Consequently, it appears that, for both cases, the scattering problem of the mode \( S_0 \) can be solved analytically to give \( R=1 \) [from Eq. (19)]. Besides, the components of the evanescent modes have to verify the homogeneous Eqs. (17) and (18). This means that for \( \nu=0 \) and for \( \Omega=\pi/\sqrt{2} \), the mode \( S_0 \) is very simply reflected by the free edge since it is not coupled to the evanescent modes. In addition, the elastic field spanned by all the evanescent modes cannot propagate toward infinity, but there may be a nontrivial solution to Eqs. (17) and (18); it is the typical situation for the existence of trapped modes which have discrete eigenvalue embedded in the continuum of scattering eigenvalues\(^23\)–\(^26\) below the first cutoff frequency of the evanescent modes. Here, for the first case, \( \nu=0 \), \( \Omega \) plays the role of the eigenvalue while, for the second case, \( \Omega=\pi/\sqrt{2} \), \( \nu \) is the eigenvalue. Usually, for a scalar case such as acoustic waveguides, the decoupling between evanescent and propagative waves is simply due to spatial symmetries between even and odd (i.e., symmetric and antisymmetric) solutions. Here, the decoupling between the propagative mode and the evanescent modes is subtler and is due to an internal\(^12\) (hidden) symmetry of the elasticity equations. For the Lamé mode the decoupling can be also physically understood by noting that, if there is no mode conversion at the horizontal boundaries, there is no more mode conversion at the vertical boundary, which is also impinged with an angle of \( \pi/4 \).

V. CONCLUDING REMARKS

We have shown the existence of a complex resonance frequency \( \Omega_k \) when the Poisson ratio \( \nu \) varies between 0 and 0.5. The real part of \( \Omega_k \) gives the frequency of the usual edge (quasi-)resonance that had been discussed in the literature; it increases monotonically from about 2 at \( \nu=0 \) up to about 2.6 at \( \nu=2.5 \). The imaginary part of \( \Omega_k \) gives the decaying rate of the resonance and has a more complicated behavior; it is zero for two values of the Poisson ratio, \( \nu_1=0 \) and \( \nu_2=0.2248 \), and decreases up to \( -2.510^{-3} \) at \( \nu=0.5 \). The two values of \( \nu \) where the complex resonance frequency has a zero imaginary part correspond to trapped modes for which there is a decoupling between the propagating waves and the evanescent waves. The first decoupling at \( \nu=\nu_1=0 \) is due to the special structure of the elasticity equations at this particular Poisson ratio value, and the second decoupling is due to the decoupling between the propagating Lamé mode and the higher order evanescent modes. Besides, it is amusing to note that, on the one hand, these two trapped modes can exist due to the decoupling between the Lamb mode \( S_0 \) and the higher order Lamb modes and, on the other hand, the edge resonance is often presented\(^8\) as a consequence of the coupling between the mode \( S_0 \) and the higher order modes (which is not false). In the future, it would be interesting to study the complex resonance frequency for more complicated edge geometries such as the beveled edge, for instance.

APPENDIX: NUMERICAL METHOD

The elasticity equations (4) and (5) can be written [see also Ref. 17]

\[
\partial_i \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}, \tag{A1}
\]

where \( \mathbf{X}=(u,v)^T, \mathbf{Y}=(-s,v)^T \), and where \( F \) and \( G \) are the operator-matrices,
\[
F = \begin{pmatrix}
-\frac{1}{\gamma} & -\frac{\gamma-2}{\gamma} \partial_y \\
\frac{\gamma-2}{\gamma} \partial_x & -\Omega^2 - 4 \frac{\gamma-1}{\gamma} \partial_x^2
\end{pmatrix}, \\
G = \begin{pmatrix}
\Omega^2 \\
-\partial_x \\
-\partial_y \\
1
\end{pmatrix}.
\]

(A2)  

(A3)

Concerning the boundary conditions (6), since the problem is considered as an evolution equation [Eq. (A1)] on \( X \) and \( Y \), we usually use the expressions of \( r \) written as a linear function of \( Y \),

\[
r \cdot Y \rightarrow r(Y) = (\gamma - 2) / \gamma s + 4(\gamma - 1) / \gamma \partial_y v,
\]

(A4)

which implies that the boundary conditions are entirely expressed in terms of \( X \) and \( Y \).

We use the differentiatation Matrix Suite proposed by Weideman and Reddy\(^{27}\) to construct the numerical method that solves the scattering problem of Sec. II. This is a spectral collocation method based on Chebyshev polynomials which yields the discretization of the differential operators \( \partial_x \) and \( \partial_y^2 \). An unknown function \( q \) is written as \( q = \sum_{n=1}^{N} \tilde{q}_n \chi_n(y) \), where \( \chi_n \) are expressed in terms of Chebyshev polynomials. Practically, in the discretization along \( y \), the function \( q \) will be replaced by the vector \( \mathbf{q} \), the first derivative \( dq/dy \) will be replaced by \( D_1 \mathbf{q} \), and \( d^2q/dy^2 \) will be replaced by \( D_2 \mathbf{q} \). Besides, with this collocation method \( \tilde{q}_n = q(y_n) \) where \( y_n = \cos((n-1)\pi/(N-1)) \).

In our problem the displacement and the stress tensor components are written as

\[
(u,v,s,t,r)^T = \sum_{n=1}^{N} (\tilde{u}_n, \tilde{v}_n, \tilde{s}_n, \tilde{t}_n, \tilde{r}_n)^T \chi_n(y).
\]

Some care has to be taken to impose the boundary conditions at \( y = \pm 1 \). One pair of conditions, \( r(\pm 1) = 0 \), yields immediately \( \tilde{r}_1 = \tilde{r}_N = 0 \). The other boundary condition, \( r(\pm 1) = 0 \), yields \( \tilde{r}_1 = \tilde{r}_N = 0 \) and needs to be translated in terms of unknowns appearing in Eq. (A1). From Eq. (A4) we have \( \tilde{s} = (\gamma - 2) \tilde{s} + 4(\gamma - 1)D_1 \tilde{v} \) which yields \( \tilde{s}_1 = 4(\gamma - 1) \tilde{v}_1 \) and \( \tilde{s}_N = 4(\gamma - 1) \tilde{v}_N \), where \( \mathbf{I}_1 \), respectively \( \mathbf{I}_N \), is the vector of the first row, respectively, last row, of the matrix \( D_1 \). Eventually the boundary conditions \( r(\pm 1) = 0 \) are written as \( \tilde{s}_1 = 4(\gamma - 1) / (\gamma - 2) \tilde{v}_1 \) and \( \tilde{s}_N = 4(\gamma - 1) / (\gamma - 2) \tilde{v}_N \). Since the boundary conditions impose the values of \( \tilde{t}_1, \tilde{t}_N, \tilde{v}_1, \) and \( \tilde{v}_N \) in the following the vectors of unknowns \( \mathbf{I} \) and \( \mathbf{S} \) are to be understood with components \( (\tilde{s}_2, \ldots, \tilde{s}_{N-1}) \) and \( (\tilde{t}_2, \ldots, \tilde{t}_{N-1}) \). With this discretization, the elasticity Eqs. (A1) become

\[
\frac{d}{dx} \begin{pmatrix}
\tilde{u} \\
\tilde{v}
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{\gamma} & M_1 \\
\frac{\gamma-2}{\gamma} D_1 & -M_2
\end{pmatrix} \begin{pmatrix}
\tilde{S} \\
\tilde{v}
\end{pmatrix},
\]

(A5)

where \( M_1 = -\frac{\gamma-2}{\gamma} D_1 - 4 \frac{\gamma-1}{\gamma} \) and \( M_2 = -\Omega^2 I - D_2 + 4 \frac{\gamma-1}{\gamma} (\mathbf{c}_l^T \mathbf{I} + \mathbf{c}_s^T \mathbf{I}) \) where \( \mathbf{c}_l \) and \( \mathbf{c}_s \) are the vector of the first column, respectively, last column, of the matrix \( D_1 \) and \( I \) is the identity matrix. Each system of ordinary differential equations (ODE) in Eqs. (A5) and (A6) is of size \( 2(N-1) \). For the sake of clarity and to keep the notation from becoming messy, all the matrices are to be understood as having the dimension corresponding to the vectors that they link together; for instance, the identity matrix in the first entry of Eq. (A5) is of dimension \( N \times (N-2) \) because it links \( d\mathbf{u}/dx \) and \( \mathbf{S} \). It remains to impose the assumption of symmetric motion on the system of ODE. First, we impose that \( N \) is an even number so that \( N = 2N_c \). Thereafter, the symmetric motion implies that \( u \) and \( s \) are even functions of \( y \) and that \( v \) and \( t \) are odd functions of \( y \), which is translated to \( u_{2N_c-n+1} = u_n, v_{2N_c-n+1} = v_n, s_{2N_c-n+1} = s_n, v_{2N_c-n+1} = -v_n, \) and \( t_{2N_c-n+1} = -t_n \) for \( 1 \leq n \leq N_c \).

At this point, the problem is discretized with respect to the coordinate \( y \) and it remains to solve the system of ordinary differential Eqs. (A5) and (A6). To do that, the matrix of this system of ODE is diagonalized, yielding \( 4(N_c-1) \) eigenvalues and eigenvectors. Actually, it is equivalent to obtain numerically the Lamb modes with the Chebyshev polynomials; there are \( 2(N_c-1) \) eigenvectors corresponding to left-going Lamb modes and \( 2(N_c-1) \) eigenvectors corresponding to right-going Lamb modes. The solution to the ODE needs the \( 4(N_c-1) \) coefficients corresponding to the \( 4(N_c-1) \) eigenvectors. The \( 2(N_c-1) \) coefficients of the numerical left-going Lamb modes are given by the radiation condition: the incident \( S_0 \) mode is given and the \( 2N_c-3 \) left-going evanescent numerical modes are set to zero. The \( 4(N_c-1) \) coefficients of the numerical right-going Lamb modes are given by the boundary condition at \( x = 0 \) that imposes \( S(x=0) = 0 \) (\( N_c \) conditions) and \( T(x=0) = 0 \) (\( N_c - 1 \) conditions). Finally, the resulting system of \( 2(N_c-1) \) equations is solved and permits one to obtain the reflection coefficient \( R \) as well as the elastic fields \( u, v, s, \) and \( t \). The complex resonance frequency is found by seeking complex \( \Omega \) where \( R \) is zero by a Newton-Raphson method.

In practice, the computations have been done with \( N_c = 20 \). The advantage of this spectral collocation method is that it is very accurate: for real frequency \( \Omega \) the conservation of energy is well respected since typically \( |R| < 1 \times 10^{-10} \) and the accuracy on the resonance frequencies \( \Omega_R \) is \( 10^{-8} \).

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Note added in proofs: During the reviewing of this paper, the author has been informed of a related study (V. Zernov, A. A. Pichugin and J. Kaplunov, Proc. R. Soc. A 2006, vol. 462, p. 1255–1270) confirming the real resonances.