On the use of leaky modes in open waveguides for the sound propagation modeling in street canyons

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An urban, U-shaped, street canyon being considered as an open waveguide in which the sound may propagate, one is interested in a multimodal approach to describe the sound propagation within. The key point in such a multimodal formalism is the choice of the basis of local transversal modes on which the acoustic field is decomposed. For a classical waveguide, with a simple and bounded cross-section, a complete orthogonal basis can be analytically obtained. The case of an open waveguide is more difficult, since no such a basis can be exhibited. However, an open resonator, as displays, for example, the U-shaped cross-section of a street, presents resonant modes with complex eigenfrequencies, owing to radiative losses. This work first presents how to numerically obtain these modes. Results of the transverse problem are also compared with solutions obtained by the finite element method with perfectly matched layers. Then, examples are treated to show how these leaky modes can be used as a basis for the modal decomposition of the sound field in a street canyon.

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I. INTRODUCTION

The aim of the present work is to solve the wave equation in a long open rectangular enclosure (Fig. 1). This open waveguide is an idealized domain of propagation modeling a street canyon. This idealized domain being a very simple representation of a real street canyon, competitive effects between guided waves (between street facades), and leaky waves (through the opening on the sky) occurring along the propagation can be more easily apprehended. In this paper, a modal based method is proposed to describe acoustic propagation in rectangular straight open waveguides with a particular attention paid to wave radiation in the free space above. Then, in other future studies in the field of urban acoustics, one could extend the principles of this method to treat more realistic street geometries.

The investigation of sound propagation in urban environments and streets has been the subject of extensive researches in the past 4 decades, as the response to a growing social demand.

After experimental observations in the 1960s, the earliest theoretical works on this topic were conducted in the 1970s. Davies, Lee and Davies, Stenackers et al., Lyon used image sources to study multiple sound reflections in a street considered as a channel between two infinite walls. Later, image source method has been improved considering scattering at facade irregularities, diffusely reflecting facades, or coherent image sources. Other energetic approaches were also used in urban acoustics. Kang developed a radiosity based model, Bradley used ray tracing method, and Picaut proposed a method based on a diffusion equation governing sound particles propagation. These energetic approaches give statistical description of sound fields in urban environments and are able to model more or less accurately numerous phenomena occurring in streets. Since these approaches assume high frequency hypothesis, they cannot describe sound fields when the wavelength is in the range of street width. Furthermore, computation costs strongly increase for complex geometries or for three-dimensional (3D) problems.

As pointed out by Lu and Walerian, interference effects are significant for relatively narrow street canyons. Hence, wave methods present real interests for sound propagation modeling. Bullen and Fricker, 30 years ago, studied the wave propagation in two-dimensional (2D) street using a modal approach, notably to model junctions of streets. To solve the wave equation in a 2D street canyon, finite element method (FEM) or boundary element method could as well give solutions to the wave equation for realistic geometries, but the high computation costs restrict their use at low frequencies or for 2D problems. The equivalent sources approach is field-based rather than ray-based, and apprehends the resonant behavior of a city canyon. In this approach, a set of equivalent sources is used to couple the free half space above the canyon to the cavity inside the canyon. The finite difference time domain (FDTD) method describes the sound field in 2D even 3D problems and can model a priori a very large number of phenomena. The parabolic equation coupled with FDTD method can also be useful to take into account meteorological effects. Even if these wave methods can model sound propagation in realistic situations, the necessary numerical resolution of the wave equation does not provide explicitly links between solution behaviors and domain geometry of the studied problem.

The aim of the present paper is to establish a multimodal description of the wave propagation in a 3D street canyon, regarded as a straight open waveguide.

In the classical case of closed waveguides having simple and bounded cross-section, a complete orthogonal modal ba-
sis \( \{ \phi_i \} \) can be analytically obtained. The case of open waveguides, as the one shown in Fig. 1, with a partially bounded cross-section, is more difficult, since no such a basis can be exhibited. However, such a cross-section being regarded as an open resonator also displays eigenmodes with complex eigenfrequencies, owing to the radiation losses.\(^{5,23-25}\)

In this paper, we propose to describe how the resonant modes of the open cross-section of an open waveguide (Fig. 1) can be used to give a multimodal formulation of the sound propagation in long open enclosures. A general method to compute the resonant frequencies and mode shapes in the transversal open cross-section of the duct is described. Results of the transverse problem are also compared with FEM computations using perfectly matched layer (PML). Then, the multimodal propagation in a straight open waveguide is formulated and numerical examples are given and discussed.

II. EIGENMODES OF THE TRANSVERSE PROBLEM

A. Theory and formulation

The wave equation to solve is

\[
\left( \nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p = 0, \tag{1}
\]

with \( \nabla^2 \) the Laplacian operator, \( c_0 \) the wave speed, and \( p \) the pressure. In a modal approach within a uniform waveguide, elementary solutions for the pressure are written as

\[
p_i(x,y,z) = e^{j(k_i x - \omega t)} \phi_i(y,z), \tag{2}
\]

where \( k_i^2 = k_0^2 - \alpha_i^2, \ Re \{ k_i \} > 0, \ Im \{ k_i \} > 0, k = \omega / c_0, \) and \( (\alpha_i, \phi_i) \) are the eigensolutions of the transverse eigenproblem

\[
\nabla^2 \phi_i - \alpha_i^2 \phi_i = 0, \tag{3}
\]

with proper boundary conditions, in the cross-section of the waveguide. Then, a solution in the waveguide can be built as a sum on these elementary solutions [the time dependence \( \exp(-j\omega t) \) is omitted]:

\[
p(x,y,z,\omega) = \sum_{i \geq 1} (a_i e^{ik_1 x} + b_i e^{-ik_2 x}) \phi_i(y,z), \tag{4}
\]

with \( i \geq 1 \) an integer number and the coefficients \( a_i \) and \( b_i \) are found as functions of the boundary conditions defined at the waveguide extremities. The transverse modes of the open waveguide shown in Fig. 1 are written as the solutions \( (\alpha, \phi) \) of the eigenproblem (3) with boundary conditions

\[
\partial_x \phi = 0 \quad \text{if} \quad z = 0 \quad \text{and} \quad |y| > L/2, \tag{5a}
\]

\[
\partial_z \phi = 0 \quad \text{if} \quad z = d \quad \text{and} \quad |y| < L/2, \tag{5b}
\]

\[
\partial_y \phi = 0 \quad \text{if} \quad y = \pm L/2 \quad \text{and} \quad 0 < z < d \tag{5c}
\]

in the cross-section of the waveguide, regarded as a 2D rectangular cavity \( \Omega_1 \) of width \( L \) and depth \( d \), open on the semi-infinite space \( \Omega_2 \) (Fig. 2).

As we present below, the modes \( \phi_i(y,z) \) can be found by solving the continuity equation at the interface between \( \Omega_1 \) and \( \Omega_2 \). A similar problem, with elastic waves, has been treated by Maradudin and Ryan.\(^{23}\) In their work, authors have calculated the discrete frequencies of the elastic vibration modes of a 2D rectangular ridge fabricated from one material that is bonded on the planar surface of a substrate of a second material. In the following, part of the equations are derived from this work to be adapted in the case of a fluid resonant open cavity.

In the cavity \( (z \geq 0) \), the general solution satisfying boundary conditions can be written as a discrete sum of functions,

\[
\phi(y,z \geq 0) = \sum_{n \geq 0} \beta_n \cos(\delta_n(z - d)) \psi_n(y) \tag{6}
\]

with \( d \) the depth of the cavity, \( \beta_n^2 = \alpha^2 - (n \pi / L)^2 \) with \( n \geq 0 \) an integer number and

\[
\psi_n(y) = \sqrt{2 - \delta_n} \cos\left( \frac{n \pi y}{L} - \frac{L}{2} \right), \tag{7}
\]

where \( \delta_n \) is the Kronecker symbol. Note that a finite impedance at the walls (absorbing material) could be considered by, e.g., modifying expression (7), although it could result in a less straightforward formulation. An alternative, then, would be to solve the transverse problem using a FEM. This would be also particularly adapted in the case of more complex cavity shapes, modeling facade irregularities.

Above the cavity \( (z \geq 0) \), the general solution is written as the spatial Fourier transform

\[
\text{FIG. 1. A straight open waveguide as model of a street canyon.}
\]

\[
\text{FIG. 2. The cross-section of a street canyon seen as a 2D open rectangular cavity.}
\]
\begin{equation}
\phi(y,z \leq 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Be^{i(\alpha y - \alpha z)} d\alpha, \tag{8}
\end{equation}

where \(\alpha_s^2 = \alpha - \alpha_i\) with \(\Im(\alpha_s) \geq 0\) and \(\Im(\alpha) \geq 0\).

Continuity equations in the interface plane \(z=0\), \(|y| < L/2\) are

\(\phi(y,z = 0^+) = \phi(y,z = 0^-)\) \tag{9}

and

\[\partial_y \phi(y,z = 0^+) = \partial_y \phi(y,z = 0^-).\] \tag{10}

A first relation between the \(\{A_n\}\) and \(B\) is found by substituting Eqs. (6) and (8) for \(\phi\) in combined Eqs. (5a) and (10):

\[\sum_{n \geq 0} A_n \beta_n \sin(\beta_n d) \psi_n(y) = -j \frac{1}{2\pi} \int_{-\infty}^{+\infty} B_\alpha e^{i\alpha y} d\alpha, \tag{11}\]

and

\[B = j \sum_{n \geq 0} A_n \beta_n \sin(\beta_n d) \frac{S_n(\alpha_s)}{\alpha_s}, \tag{12}\]

where

\[S_n(\alpha_s) = \int_{-L/2}^{+L/2} \psi_n(y) e^{-i\alpha y} dy. \tag{13}\]

A second relation is found by using the continuity equation (9):

\[\sum_{n \geq 0} A_n \cos(\beta_n d) \psi_n(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Be^{i\alpha y} d\alpha; \tag{14}\]

whence it follows that

\[A_m \cos(\beta_m d) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} BS_m^s(\alpha_s) d\alpha_s, \tag{15}\]

where \(m \geq 0\) is an integer. Then, Eqs. (12) and (15) lead to the set of linear, homogeneous equations (16) for the \(\{A_n\}\):

\[\forall m \in \mathbb{N}, \quad A_m \cos(\beta_m d) = j \sum_{n \geq 0} \Pi_{mn}(\alpha) \beta_n A_n \sin(\beta_n d), \tag{16}\]

where

\[\Pi_{mn} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{S_n^s S_m^s}{\alpha_s} d\alpha_s. \tag{17}\]

It is easily shown that, for real values of \(\alpha_s\), \(\Pi_{mn}\) vanishes unless \(m\) and \(n\) have the same parity. \(23\) Then, Eq. (16) breaks up into two sets of linear equations, one governing the symmetrical eigenmodes (even functions of \(y\) with even values of \(m\) and \(n\)), and the other governing antisymmetrical eigenmodes (odd functions of \(y\) with odd values of \(m\) and \(n\)).

The following equation gives a general expression of \(\Pi_{mn}\) for both even or odd indices \(m\) and \(n\):

\begin{equation}
\pi_{mn}(\alpha) = (-1)^{(m+n+2)} \frac{2 - \delta_{m0}}{2} \frac{2 - \delta_{n0}}{2} \alpha_s^2 \int_{0}^{+\infty} \alpha_s \left( \frac{\alpha_s + m \pi}{L} \right) \left( \frac{\alpha_s + n \pi}{L} \right) \times \text{sinc} \left( \frac{\alpha_s - m \pi}{2} \right) \text{sinc} \left( \frac{\alpha_s - n \pi}{2} \right) d\alpha_s. \tag{18}\end{equation}

Finally, Eq. (16) can be written in the matrix form

\[D \tilde{A} = \tilde{0}, \tag{19}\]

where the components of vector \(\tilde{A}\) are \(A_n \sin(\beta_n d)\) and terms of matrix \(D\) are

\[D_{mn}(\alpha) = \cot(\beta_m d) \delta_{mn} - j \Pi_{mn}(\alpha_s). \tag{20}\]

Then, the eigenvalues of the transverse eigenproblem are the values \(\alpha_i\) of \(\alpha\) for which \(\det(D) = 0\), and the eigenfunctions \(\phi_i\) are given by the corresponding set of coefficients \(A_n^{(i)}\), satisfying Eq. (19) with \(D = D(\alpha_i)\).

\section{B. Numerical resolution}

After truncation of Eq. (19) at a finite-size matrix problem, zeros of the determinant of \(D\) are numerically located in the complex \(\alpha\)-plane to compute eigenvalues \(\alpha_i\) of the transverse problem (Fig. 3).
Owing to the radiation losses in the infinite space $\Omega_2$ (above the waveguide), eigenvalues are complex, lying in the lower half-plane.\textsuperscript{3,23–25} The spectrum displays families of eigenvalues corresponding to either symmetric (blue circles $\bigcirc$) or antisymmetric modes (red crosses $\times$). An analogy with the classical, real, modes of the simple problem with a Dirichlet condition ($f=0$) at $z=0$ instead of the exact radiating condition written in Sec. II A, allows us to label the complex modes $\phi_i$—at least the modes which eigenvalue is located close enough to the real axis in the complex $\alpha$-plane—with a couple of integers denoting the number of vertical and horizontal nodal lines (Fig. 4). Following this terminology, the families displayed in Fig. 3 are the $\phi_{(p,q)}$ with $p$ constant.

Figure 3 shows the spectrum of eigenvalues for two different values of the aspect ratio of the cavity: (a) $d/L=1.4$ and (b) $d/L=2$. The pattern in both plots is similar, exhibiting the families of modes $\phi_{(p,q)}$ with $p$ constant. However, in the “deeper” cavity, with the aspect ratio $d/L=2$, the confinement of the modes is more important; thus, eigenvalues $\alpha_i$ have a smaller imaginary part than in the cavity with aspect ratio 1.4.

For comparison, a finite element method is used to solve the transverse eigenproblem (3). The semi-infinite space above the cavity is bounded with PMLs, as used by Hein et al.\textsuperscript{26} and Koch\textsuperscript{5} in a similar problem (Fig. 5).

The results shown in this paper (Figs. 3(b) and 6) have been obtained with parameters $d'=L/2, l'=L, d_{PML}=L/2$, and $\tau_y=\tau_z=1+j$ (see Appendix). A Dirichlet condition $f=0$ is imposed on the outer boundaries of the PML. Moreover, as the geometry of the cross-section is symmetric about the $z$ axis, one-half only of the domain is meshed, with the appropriate symmetry or antisymmetry condition imposed at $y=0$. Computations have been performed using the partial differential equation toolbox from MATLAB\textsuperscript{6}.

The results of the two compared methods—the resolution of Eq. (19) and the FEM—are in good agreement [Fig. 3(b)]. The discrepancy between the results increases for larger values of the imaginary part $\Im\{\alpha_i\}$. However, eigenvalues that are less well-estimated are associated with modes that will be strongly attenuated when propagated in the waveguide. They are then of secondary importance when considering the transport of energy on a sufficiently long distance. Moreover, it will be shown in the following that the contribution of these modes in the determination of the sets of coefficients $\{a_i\}$ and $\{b_i\}$ is almost negligible.

A comparison between the eigenfunctions $\phi_i$ deduced from the method detailed in Sec. II A and from the FEM computation shows also a good agreement (Fig. 6). Both methods give very similar results, even when the error on the imaginary part of $\alpha_i$ becomes more significant (Fig. 6, bottom). The figure also clearly shows that low order modes weakly “radiate” in the infinite space $\Omega_2$; the effect of the opening of the waveguide appears as a small perturbation on the classical, nonradiating, solutions that would be obtained by applying a homogeneous Dirichlet condition at the top of the waveguide ($z=0$). For higher order modes, however, eigenfunctions $\phi_i$ differ more and more from the real, “Dirichlet,” solutions. Patterns of nodal lines are more complex. Consequently, the indexation with indices $(p,q)$ become less relevant, and the confinement, which was strong for low order modes, becomes weaker, with the energy increasing near the interface $z=0$.

Now that the transverse eigenmodes are determined, they can be used to write a multimodal formulation of the sound propagation within the open waveguide that displays a street canyon.

\section*{III. Propagation Along the Street}

As explained in Sec. I, the transverse modes $\phi_i$ are used, for given source and radiation conditions at the ends of the waveguide, to built a solution of the wave equation, as written in Eq. (4).

Because the transverse eigenvalues $\alpha_i$ are complex with $\Im\{\alpha_i\}<0$, the propagation constants $k_i$ are also complex, with $\Im\{k_i\}>0$, even for real source frequency $\omega$. Then, all the modes $\phi_i \exp(\pm jk_i x)$ in the waveguide decrease exponentially while propagating, reflecting the radiation losses during the propagation along the open waveguide. This corresponds to leaky modes.

Then, the two sets of coefficients $\{a_i\}$ and $\{b_i\}$ must be found, as functions of the end conditions in the waveguide. At the input end of the waveguide, a source condition is defined as a given acoustic pressure distribution in a plane $x=\text{const}$, with frequency $\omega$. For example, at $x=0$: $p(0,y,z,\omega)=p_0(y,z)\exp(-j\omega t)$. At the output end of the waveguide is given a radiation condition.
Thus, after truncation at a finite number

A. Input condition

Let us call $P_i(x) = a_i \exp(jk_i x) + b_i \exp(-jk_i x)$ the coefficients in the development in series (4). Since the modes $\phi_i$ are not orthogonal, the initial field $p_0(y, z, \omega)$ cannot be projected on the $\{ \phi_i \}$ as it is classically made to find the $\{ P_i(0) \}$. Thus, after truncation at a finite number $N$ of terms in the development (4), a least squares method is thus used to find these coefficients:

$$\tilde{P}(0) = (\Lambda^{(\phi)})^{-1} \tilde{p}_0^{(\phi)}, \quad (21)$$

where the $i$th component of $\tilde{P}(0)$ is $P_i(0)$ and

$$\Lambda^{(\phi)}_{ij} = \langle \phi_i | \phi_j \rangle, \quad (22a)$$

$$p_{0i}^{(\phi)} = \langle \phi_i | p_0 \rangle, \quad (22b)$$

with the product

FIG. 6. Antisymmetric eigenfunctions $\phi_{3,m}$. $q > 0$, and comparison with FEM computations. The PML domain is not shown in the FEM results.
\[ \langle f|g \rangle = \int \int f g d\Omega. \] (23)

Practically, we consider source conditions \( p_d(y,z,\omega) \) with a compact support, included in \( \Omega_1 \), i.e., inside the street canyon. Thus, for convenience when determining \( \vec{P}(0) \), we consider, as the eigenfunctions \( \phi_i \), their restriction to the domain \( \Omega_1 \), that is,

\[ \forall (y,z) \in \left[ -\frac{L}{2}, \frac{L}{2} \right] \times [0,d], \]

\[ \phi_i(y,z) = \sum_{n=0} A_n^i \cos(\beta_n^i(d-z))\psi_n(y), \] (24)

and the product \( \langle f| \) above is

\[ \langle f|g \rangle = \int_0^d \int_{-L/2}^{L/2} f g dydz. \] (25)

Note that in the case of real orthogonal modes, as in classical closed waveguides, the least squares method gives the usual projection coefficients \( P_m = \langle \phi_m|p \rangle \).

### B. Output condition

Let \( Q(x) = jk_0 \sum_{\gamma} e^{j k_0 x} \) be the coefficients in the development of the \( x \)-component of \( \nabla p \). One assumes that at the output end of the waveguide, say, at \( x = x_{end} \), the condition is given as an admittance matrix \( Y_{end} \) fulfilling \( \vec{Q}(x_{end}) = Y_{end} \vec{P}(x_{end}) \).

Again, as for the formulation at the input end, and due to the non-orthogonality of the eigenmodes, the matrix \( Y_{end} \), for some complex end conditions, may not be straightforwardly calculated. However, usual end conditions—rigid end, non-radiating open end, anechoic termination—can be easily formulated with this type of admittance matrix, generalization for all modes of the usual admittance for the plane wave.\(^{28}\)

### C. Solutions for \( \{a_i\} \) and \( \{b_i\} \)

Now that an input condition \( \vec{P}(0) \) and an output condition \( Y_{end} \) are known, the vectors \( \vec{a} \) and \( \vec{b} \) of the \( \{a_i\} \) and \( \{b_i\} \) in Eq. (4) can be calculated:\(^{29,28}\)

\[ \vec{a} = (1 - \delta)^{-1} \vec{P}(0), \] (26)

\[ \vec{b} = -\delta(1 - \delta)^{-1} \vec{P}(0), \] (27)

where \( \delta = D_1 Y_{end} + Y_{end} D_1 \), \( D_1 \) is diagonal with terms \( D_{1,i} = \exp(j k_0 x_{end}) \), \( Y_{end} \) is diagonal with terms \( Y_{i,j} = j k_0 \).

Thus, with these solutions for \( \vec{a} \) and \( \vec{b} \), the pressure field in the open waveguide can be calculated. However, terms \( \exp(-j k_0 x) \) can be the source of numerical problems of convergence. Then, defining \( \vec{b} = D_1^{-1} \vec{b} \), the pressure field is written as

\[ p^{(N)}(x,y,z,\omega) = \sum_{i=1}^{N} (a_i e^{jk_0 x} + b_i e^{jk_0 (x_{end} - x)}) \phi_i(y,z). \] (28)

This new formulation depends only on \( D_1 \), not on \( D_1^{-1} \), and on exponentials with positive arguments \( x \) or \( L-x \).\(^{30}\)

![FIG. 7. Top: Initial condition at abscissa \( x=0 \). The parameters of the three Gaussian functions in Eq. (29) are \( (y_1, z_1, \omega_1) = (0.18L, 0.42d, 0.24) \), \( (y_2, z_2, \omega_2) = (-0.30L, 0.26d, 0.22) \), and \( (y_3, z_3, \omega_3) = (-0.14L, 0.70d, 0.24) \). Bottom: Modal reconstruction of the initial condition \( p_0(y,z) \), using \( N = N' = 30 \) modes.](Image)

### IV. RESULTS

In the following, for simplicity, we will consider the wave field downstream from a source in an infinite waveguide. Then, there are no back propagated waves: \( b_i = 0 \).

#### A. Input condition

The initial condition at \( x=0 \) is the pressure distribution shown in Fig. 7 (left part) and given by

\[ p_0(y,z) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{3} e^{-((y-y_k)^2 + (z-z_k)^2)/2 \sigma_k^2}, \] (29)

where \( \sigma_k \in \mathbb{R}^+ \) and \( (y_k, z_k) \in [-L/2, L/2] \times [0,d] \). The associated dimensionless frequency is \( k L/2 \pi = 1.2 \). This input condition is chosen as a nontrivial solution for the modal formulation. Moreover, it can be seen as a simple and general way to describe several, spatially distributed, sources.

Then, the \( \{a_i\} \) are found by substituting Eq. (29) for \( p_0(y,z) \) in Eqs. (21) and (22). The modal reconstruction is shown in Fig. 7. Using a basis of \( N = N' = 30 \) modes we recall that \( N \) is the number of terms in the series (28) and \( N' \) is the size of the linear problem (21) in the least square estimation of the \( \{a_i\} \), so that \( N = N' \), the input pressure condition is well reproduced with a residual error of 3.6\%. This error is due to the high order depth modes of the first families that have deliberately not been considered in the modal basis because of their weak of relevance in the propagation. Furthermore, it will be shown in the following that omitting these modes does not affect significantly the estimation of the \( \{a_i\} \) for the modes taken into account.

To evaluate the convergence of the method when increasing the number \( N \) of modes taken into account in Eq. (28), from a \( N = 30 \) modes basis, an error \( \varepsilon \) is defined as

\[ \varepsilon = \sqrt{\frac{1}{d} \int_0^d \int_{-L/2}^{L/2} ||p^{(N)} - p_0||^2 dydz}, \] (30)

where \( p^{(N)} \) is the modal solution obtained with \( N \) modes [Eq. (28)], and \( p_0 \) the reference field. The modes are sorted by
increasing value of the imaginary part of their propagation constant \( k_n \), that is, from the least damped to the most damped leaky mode propagating along the canyon (Table I). This type of classification depends directly on each eigenvalue and the source frequency.

The convergence of the reconstruction of the initial condition \( p_0(y,z) \) is shown in Fig. 8 (circles \( \bigcirc \)). Naturally in such a modal approach, depending on the source distribution, each mode introduced in the computation contributes differently to the reconstruction of \( p_0(y,z) \).

### B. Propagation within the waveguide

Assuming that the convergence is reached for \( N=30 \), the field \( p^{(30)}(x,y,z,\omega) \) is now taken as the reference field to compute an estimation error at abscissa \( x=10L \) (triangles \( \bigtriangleup \) in Fig. 8). The variability, depending on the initial condition \( p_0(y,z) \), of the contributions of the modes to the convergence is still visible, but, moreover, it clearly appears that only a few modes—the less damped modes—still contribute to the transport of energy at that distance from the “source.” Practically, sorting the modes as done in Table I is thus a good choice to increase the convergence, as soon as one is interested by the wave field in the street canyon at a sufficient distance from the source.

Figure 9 shows the field in the cross-section of the waveguide at abscissa \( x=10L \), \( L \) the width of the waveguide. The left plot is obtained using Eq. (28) and \( N=30 \)—the total number of modes used to perform the least squares estimation, while the right plot is obtained using only the first six modes, with the ordering defined above. Both results are very close: the relative error between them, defined as in Eq. (30), is less than 0.35%.

Since only a few modes are necessary to describe the field at certain distance from the source, it would be advantageous to use a reduced modal basis in the computation. Since modes are not orthogonal, the value of each modal coefficient \( a_i \) depends on the size of the “basis” used in the least squares estimation. One shows, however, that this dependency is rapidly weak, in particular, for the first modes, that is, the less damped (Fig. 10). The notation \( a_i^{(N)} \) is used to denote the number of modes \( N \) taken into account for the least squares estimation of \( a_i \).

To evaluate the relevance of using a reduced modal basis, two modal solutions of the wave equation in the infinite street canyon with the initial condition (29) at \( x=0 \) are compared: the solution with \( N=N=10 \), and the solution with \( N=30 \).
and radiation of the wave in such partially bounded geometries can be numerically determined. As these are complex modes that decay exponentially while propagating, the number of modes that effectively carry the wave field (emitted by some source in the waveguide) decreases rapidly, so that only a few modes, at a sufficiently large distance from the source, is necessary to accurately model the wave propagation. This gives this approach a real interest for numerical computations, in addition to its interest as a physically meaningful description of the street as a partially confining and guiding medium for the acoustic waves. As a first step and to clearly point out the principles and interests of our approach, only a uniform open waveguide was considered in this paper. Following, cross-section discontinuities, as model of the junctions between buildings, can be considered by using mode matching techniques.

APPENDIX: FORMULATION IN THE PERFECTLY MATCHED LAYERS

PML are used as a tool to avoid non-physical reflections at the boundaries of a necessarily finite domain in a numerical computation. The method works as follows: the solution $\phi(y,z)$ of the eigenproblem (3) above the cavity is analytically continued in the PML with respect to variables $(y,z)$ to complex variables $(\hat{y}, \hat{z})$. The extended solution $\tilde{\phi}$ satisfies

$$\left(\frac{\partial^2}{\partial \hat{y}^2} + \frac{\partial^2}{\partial \hat{z}^2}\right) \tilde{\phi} = 0. \quad (A1)$$

Complex variables $(\hat{y}, \hat{z})$ are now written as

$$\hat{y}(y) = \int_{y'}^y \tau_y(y')dy', \quad \hat{z}(z) = \int_{z'}^z \tau_z(z')dz' \quad (A2)$$

with $\Re\{\tau_{y}\} > 0$, $\Im\{\tau_{y}\} > 0$, and $\tau_y(y \leq y') = 1$, $\tau_z(z \leq z') = 1$ (Fig. 5). The results in this paper have been obtained with $\tau_y(y > y') = \tau_z(z > z') = 1 + j$.