Non-cascade frequency-mixing processes for elastic waves in unconsolidated granular materials

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Abstract

Due to disorder, contacts between beads in stressed granular assemblages are not equally loaded. There exists a distribution of contact static loads, containing weakly loaded contacts (the weak contacts) and strongly loaded contacts (the strong contacts involved in the so-called force chains). For an elastic periodic excitation with a given deformation amplitude, the weakest contacts are supposed to clap (periodically open and close) due to the action of the acoustic wave. When increasing the acoustic amplitude, more and more contacts are clapping, progressively producing a non classical spectral signature.

Presented spectra have been observed in a laboratory scale experiment, where two frequencies were initially launched in the medium. Results are obtained for increasing pump wave amplitudes and different frequency pairs. These experimental results are in good agreement with a model derived from the Hertz theory of contacts with possibility of clapping.

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1. Introduction

In acoustics of homogeneous materials, the broadening of the spectrum of a weak signal launched in the medium could be viewed as a sequence of nonlinear wave mixing processes. Higher combinational frequencies could be imagined as being generated through a few subsequent nonlinear scattering processes of the lowest order rather than through a single nonlinear scattering of higher order. This is due to smoothness of the dependence of strain on stress and its successive derivatives in gas, fluids and homogeneous solids. In theory, it is sufficient to use the first nonlinear term in the Taylor expansion of the stress–strain relationship for adequate description of weakly nonlinear acoustic waves.

In micro-inhomogeneous solids, the situation could be different because of possible abrupt changes in stress–strain relationship. In particular, in such materials as rocks, exhibiting hysteretic nonlinearity [1], abrupt changes in stress–strain relationship take place in transition from loading to unloading. In the precise description of stress–strain relationships, which exhibit discontinuous features themselves or discontinuous features in their derivatives, multiple terms of the Taylor expansion could be important. Thus higher order non linear phonon scattering processes could be of comparable importance with the lowest order processes. The highest combinational frequencies can be efficiently generated simultaneously with the lowest ones, that is through a non-cascade single scattering process involving the necessary number of phonons.

Non-cascade processes are also known to be efficient for the so-called contact nonlinearity which could be realized in opening/closing the gaps in unperfect elastic contacts between two surfaces [2,3]. Under periodic loading, the contacts between surfaces or between the lips of cracks could move in a regime of clapping/tapping, which is characterized by non smoothness of the local stress–strain relationship and/or its derivatives. Recently, the importance of the nonlinear phenomena induced by intermittent
contacts between grains has been indicated in the case of unconsolidated granular media for the self-demodulation process [4], the harmonic generation [5] and the subharmonic and noise excitation [6,7]. The nonlinear process of modulation transfer (Luxembourg–Gorky) has also been used for the monitoring of small perturbations of the granular assemblage state [8]. In this case, the small induced mechanical perturbations modify mostly the weakest contacts in the medium, which has a strong influence on the detected nonlinear elastic response of the medium.

However, from the non-cascade processes, only the process of harmonic generation has been mostly studied [2]. In this communication we present our research results on the process of harmonic generation has been mostly studied [2]. In this communication we present our research results on the process of harmonic generation [2], the harmonic generation [3], and the subharmonic and noise excitation [4]. In this communication we present our research results on the process of harmonic generation has been mostly studied [2].

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2. Experimental results

The experimental setup comprises a container 8 cm × 8 cm × 8 cm, filled with monodisperse glass beads of 2 mm-diameter. The acoustic probing is performed by identical wide-band piezo-emitter and receiver, placed on opposite side walls close to the bottom of the container. At this depth, the granular medium being only stressed by its own weight, the average static stress is less than 3 kPa, which approximately corresponds to an average contact deformation of 3 × 10⁻⁵ (the maximum strain excitation amplitude, 10⁻⁵ remains less than this value but close). The performed experiment consists in launching in the medium a two-frequency component signal (the pump wave) and to observe the spectrum as a function of the excitation amplitude. In Fig. 1, such spectrum is plotted for three excitation amplitudes.

This experiment is carried out using a spectrum analyzer (Stanford SR830) both for the generation of the two-frequency component signal and for the spectrum analysis. The amplification of the excitation signal is performed via a Bruel & Kjaer low distortion amplifier to a maximum of 140 Vrms at the output. The received signal feeds directly the spectrum analyzer. Carrying exactly the same experiment in air gives, for the maximum excitation amplitudes, a small contribution (under −95 dB) at the demodulated frequency (possibly due to the setup small nonlinearity but also coming from the classical nonlinear effects in air). No other frequencies (and in particular the observed non-cascade frequency components) can be detected in this test experiment carried out in air. Components denoted here as the non-cascade frequency components, could be in principle generated through a cascade of single scattering events, but this would have to involve a very large number of these successive events.

It is possible to observe, for the weakest excitation amplitude $\varepsilon_A \approx 7 \times 10^{-7}$, that only the two excitation frequencies $f_1 = 35$ kHz and $f_2 = 45$ kHz are present in the detected spectrum. For $\varepsilon_A \approx 4 \times 10^{-6}$, the difference frequency $f_2 - f_1$, classically obtained in the first nonlinear cascade process emerges from electronic noise level. At the highest amplitude $\varepsilon_A \approx 10^{-5}$, other frequencies are visible, 5, 15, 20 and 25 kHz, as well as a continuous noise spectrum for the low frequencies. The overall spectrum is controlled by the frequency response function of the medium and the transducers presented in Fig. 2. It can be seen that the shape of the noise spectrum in Fig. 1 corresponds to the frequency response function, i.e., even if all the frequencies are generated by the nonlinear process of acoustic noise excitation, only the ones efficiently transmitted and detected can be observed. This feature has been already observed and interpreted in [6]. It is possible to define a cut-off frequency of the medium, above which the waves are highly attenuated (mostly by multiple scattering due to the disordered packing or by the band gap effects in regular packings). This is also the reason why, frequency-up conversions are not easily observed in this experiment.

In Fig. 3, the amplitude dynamics of several frequency components as a function of the excitation amplitude are plotted. The pump frequency 35 kHz has a linear dependence, while the difference frequency component at...
10 kHz exhibits a behavior close to a quadratic one. For the components that can not be obtained in the first cascade process, the dynamics is different, with strong slope variations (5 kHz and 15 kHz in Fig. 3). The observed corresponding amplitudes are well above the electronic noise level for the highest excitation amplitudes. However, they exhibit some small variations at time scales of the order of the second, and quantitative conclusions on the amplitude dynamics are consequently hard to draw.

These behaviors have not been observed for cascade processes in media with weak nonlinearity and are theoretically hard to draw.

The results presented in Figs. 1 and 3 are obtained for a given excitation frequency pair \((f_1 = 35 \text{ kHz and } f_2 = 45 \text{ kHz})\). However, spectral components, and especially the non-cascade ones, depend strongly on the chosen frequency pair of excitation, as explained in the next section.

This is visible in Fig. 4, which presents a large number of data for different excitation frequency pairs. Each row corresponds to a spectrum where amplitudes are coded in colors. In Fig. 4a, \(f_1\) is fixed at 30 kHz while \(f_2\) is varied from 40 to 50 kHz. In Fig. 4b, \(f_1\) and \(f_2\) are varied from 30 to 40 kHz and from 40 to 50 kHz respectively. In Fig. 4c, \(f_1\) is varied from 30 to 40 kHz while \(f_2\) is fixed at 40 kHz. It is important to note that lower frequencies than the difference one emerges from noise, and some patterns can be observed (Fig. 4b).

These observed qualitative features are explained in the next section, with the help of a simple model for a particular non smooth nonlinearity.

3. Theoretical analysis

In the following, a simple analysis of the observed spectra is performed, taking into account the presence of a clapping type nonlinearity at the level of the contacts. The clapping type nonlinearity and the hysteretic nonlinearity contain singularities in their stress–strain relationships or their successive derivatives. It is expected that the qualitative features of such non smooth nonlinearities are similar in the effects observed here. Taking into account the estimates for the generation and amplification of subharmonics in a medium with quadratic nonlinearity [6], nonlinearities with singularities (clapping, tapping, hysteretic) are the most relevant to explain the observed effects of subharmonic and noise excitation at such low excitation levels.

Let us consider for instance the static stress–strain relationship of an Hertzian contact of the form

\[
\sigma_0 = -C(-\varepsilon_0)^{3/2},
\]

where \(\sigma_0\) is the static stress, \(\varepsilon_0\) is the strain and \(C\) a constant related to the elastic parameters and the radii of the spheres [9]. When an acoustic wave excites the contact, the dynamic stress–strain relationship can be written as [4]

\[
\sigma_0 + \tilde{\sigma} = -C(-\varepsilon_0 - \tilde{\varepsilon})^{1/2}H(-\varepsilon_0 - \tilde{\varepsilon}),
\]

with the dynamic part of the stress \(\tilde{\sigma}\), the dynamic part of the strain \(\tilde{\varepsilon}\) and the Heaviside function \(H(-\varepsilon_0 - \tilde{\varepsilon})\) which takes into account the fact that there is no stress between the beads when they are not in contact.

In the case where \(|\tilde{\varepsilon}| \ll |\varepsilon_0|\), the stress–strain relationship can be expanded in Taylor series and the dominant nonlinearity is the classical quadratic nonlinearity [4]. In the case when the previous strong inequality is not fulfilled, the expansion is no more possible and if \(|\tilde{\varepsilon}| > |\varepsilon_0|\), the Heaviside function begins to play an important role in the nonlinearity.

At first, by introducing a strain signal \(\tilde{\varepsilon}\) composed of two frequencies \(f_1\) and \(f_2\), it is possible to calculate by fast Fourier transform, the resulting stress \(\tilde{\sigma}\), using Eq. (2). With the same frequencies as in the experiment in Figs. 1 and 3, \(f_1 = 35 \text{ kHz and } f_2 = 45 \text{ kHz}\), the result is plotted.
for different excitation amplitudes in Fig. 5. The same frequencies as those observed experimentally in Fig. 1 are obtained, including the non classical ones at 5, 15, 20, and 25 kHz for instance. The main differences between this calculated spectrum and the experimental one are associated with the influence of the experimental frequency response function of the system (see Fig. 2).

In Fig. 6, the amplitude dynamics of some particular frequency components of the spectrum are plotted as a function of the excitation amplitude. Similarly to the experimental amplitude dynamics of Fig. 3, the dynamics of the pump and difference frequencies (35 and 10 kHz) are monotonous and the dynamics of the 5 and 15 kHz components has a more complex behavior in amplitude, with strong slope variations.

In order to analyze theoretically the non classical role of the Heaviside function in the stress–strain relationship, let us consider, for example, a two-dimensional granular medium composed of cylinders. For small loadings, the contact between two cylinders is known to provide a stress proportional to the strain (linear elasticity) if the contact is closed [9]. However, the open contact is unstressed. As a result, the considered micromechanical element is strongly nonlinear. The corresponding stress–strain relationship could be described in this case by

$$\sigma = C e H(-e).$$

(3)

The stress–strain relationship (3) is schematically plotted in Fig. 7. Note that in our notation, negative strain (compression) corresponds to a closed contact. There is a difference between Eq. (3) and (2), namely the presence of the power $3/2$ due to the Hertz contact between two beads in Eq. (3) and the linear dependence in Eq. (2). The simpler bilinear case of Eq. (3) enables to perform analytical calculations on the frequency-mixing processes associated with the Heavyside function in the stress–strain relationship. Numerical calculations of spectra of the form Fig. 5 in the case (3) and in the case (2) have been perform and do not reveal qualitative differences in the emerging frequency components. The power $(3/2$ or 1) in the stress–strain relationship influences mainly the amplitude dynamics of the frequency components of the spectrum.

In the dynamics of contacts, two regimes are commonly distinguished in the literature [2]. The clapping regime corresponds to an initially (in the absence of acoustic field) closed contact $\varepsilon_0 < 0$ and the tapping regime corresponds to an initially open contact $\varepsilon_0 > 0$ [2].

The strain at the individual contact between two cylinders can be written,

$$\varepsilon = \varepsilon_0 + \varepsilon_1 \cos(\omega_1 t) + \varepsilon_2 \cos(\omega_2 t + \phi),$$

(4)

where $\varepsilon_{1,2}$ are acoustic strain amplitudes and $\phi$ is a phase term. According to (3) and Fig. 7, the response of the system to the applied strain is linear with increasing $\varepsilon_{1,2}$ until the critical pump amplitude is such that $\varepsilon_1 \cos(\omega_1 t) + \varepsilon_2 \cos(\omega_2 t + \phi)$ reaches the value $-\varepsilon_0$. From this critical pump wave amplitude, the contact starts to open during
a period $T_0 = t_2 - t_1$ of its motion between characteristic times $t_1$ and $t_2$ (see inset in Fig. 8). The acoustic strains higher than $-\omega_0$ are not transmitted by the contact. This strong threshold nonlinearity influences the spectrum of the stress quite differently than weak nonlinearity.

In Fig. 8, the strain excitation $\varepsilon$ is plotted as a function of time. The corresponding stress dynamic evolution deduced from this strain excitation and from the stress–strain relationship is plotted in Fig. 9.

The stress $\sigma$ can be decomposed into a linear contribution $\sigma_1$, which does not modify the spectrum, and a nonlinear contribution $\sigma_{nl}$, which is responsible for the signal modification,

$$
\sigma = \sigma_1 + \sigma_{nl} = E\varepsilon - E\varepsilon[1 - H(\varepsilon)].
$$

The contribution $\sigma_{nl}$ exists only in the interval $t_1 \leq t \leq t_2$ near the maximum of the strain signal at $t = t_0$ (see Fig. 8).

In the case plotted in Fig. 9, the single maximum over the period $-T/2 \rightarrow T/2$ is characterized by its central time occurrence $t_0$. For the particular case where two frequencies $f_1$ and $f_2$ are initially launched in the medium, the total period of the signal (4) is equal to the inverse of the greatest common divisor of these frequencies, i.e., if $T_1 = 1/f_1$, $T_2 = 1/f_2$ then the signal period is $T = n_2T_2 = n_1T_1$ where $n_1$ and $n_2$ are the smallest possible integers.

Importantly, the localized character of the nonlinear contribution to the stress $\sigma_{nl}$ with a time scale duration $T_0 = t_2 - t_1 \ll T$ near the central time $t_0$ ensures that the strong nonlinearity excites simultaneously (in a single act of nonlinear scattering) many new spectral components at $F = m/T$ where $m = 0, 1, 2, \ldots$ which can be different from the frequencies obtained in the first step of successive approximations (namely $f_2 - f_1$, $f_1 + f_2$, $2f_1$, $2f_2$) because

$F$ can be different from $f_2 - f_1$. For instance, the period of signal $\cos(2\pi f_1 t) + \cos(2\pi f_2 t)$ with $f_1 = 31$ kHz and $f_2 = 41$ kHz is $T = 31/(31 \text{ kHz}) = 41/(41 \text{ kHz}) = 1/1000 \text{ s}$ which is 10 times longer than the period of the difference frequency component at 41 kHz - 31 kHz = 10 kHz.

If the excited strain is written in the form (4), it is possible for $\phi = 0$, by looking for the extremum close to $t = 0$ (zero of the time derivative of the total strain), to find an expression for the characteristic time $t_0$ as

$$
t_0 \simeq -\frac{\varepsilon_0\omega_2}{\varepsilon_1\omega_1^2 + \varepsilon_2\omega_2^2}\phi. \quad (6)
$$

The complex amplitudes of the nonlinear stress components at frequencies $F = mf$ are denoted by $\sigma_{nl}$ and can be written as

$$
\sigma_{nl} = \int_{-T/2}^{T/2} \sigma_{nl} e^{i2\omega t} \, dt = \int_{t_1}^{t_2} \sigma_{nl} e^{i2\omega t} \, dt. \quad (7)
$$

Under the condition $m < \frac{1}{2\pi(\omega_1 - \omega_2)}$ the complex amplitude of the nonlinear stress excited in the clapping process at the frequency $F = mf$ can be written as

$$
\sigma_{nl} \simeq e^{i2\omega f_0} \int_{t_1}^{t_2} \sigma_{nl} \, dt = S_\sigma e^{i2\omega f_0}, \quad (8)
$$

where $S_\sigma$ is the area of the stress which is cut by the singularity of the stress–strain relationship (see Fig. 8).

From (6) and (8), the excitation phase of the frequency $mf$ is deduced:

$$
\phi_{mf} \simeq mf \frac{\varepsilon_1f_2}{\varepsilon_1f_1 + \varepsilon_2f_2}\phi. \quad (9)
$$

These results show that frequencies $mf$ (with $m < \frac{1}{2\pi(\omega_1 - \omega_2)}$) are excited with equal amplitudes in a single act of scattering, which is different from a classical cascade process. The phase of the $mf$ frequency excitation depends
on the relative amplitude of the excited waves at $f_1$ and $f_2$. This last point could be checked experimentally in the future. However, a dependence of the non-cascade frequency component amplitude on the relative phase $\phi$ has been experimentally observed. The non monotinous dynamics can be explained by the occurrence of more and more saturated extrema in the stress when the excitation amplitude is increased. Their appearance excites a given non-cascade frequency, sometimes in phase with previously existing extrema or sometimes out of phase. This is an opportunity to explain the observed complex behavior in Fig. 3 for instance. Looking in details at the frequencies present in each spectrum of Fig. 4, obtained for different excitation frequency pairs, the expectation from the simple developed model that frequencies $F = m/T$ are excited is confirmed. As an example, for $f_1 = 31 \text{ kHz}$ and $f_2 = 41 \text{ kHz}$ (corresponding to $1/T = 1 \text{ kHz}$ while the frequency of the pump signal envelope and thus the demodulated frequency is $10 \text{ kHz}$), frequencies 2, 5, 6, 7, 9, 10 and 11 kHz can be observed. For $f_1 = 32 \text{ kHz}$ and $f_2 = 42 \text{ kHz}$ (corresponding to $1/T = 2 \text{ kHz}$), frequencies 4, 6, 8, 10 and 12 kHz are observed. This corresponds well to the $m/T$ frequencies. It remains however, in each experiment where the difference frequency is $10 \text{ kHz}$, a frequency of $5 \text{ kHz}$ which can not be explained everytime by this simple analysis. This could be associated with period doubling effects for the difference frequency wave itself [6].

4. Conclusions

A non-cascade nonlinear process of frequency mixing has been studied in experiments and theory. Due to the singularity of the stress–strain relationship at the level of a single contact, the nonlinearity becomes of infinite order and higher order nonlinear phonon scattering processes are of comparable importance with the lowest order processes. One of the peculiar signatures of high order nonlinearity is the possible excitation of arbitrarily low frequencies from two well-chosen excitation frequencies, because the lowest excited frequency is equal to the greatest common divisor of these two frequencies. The nonlinear acoustic resolution of the pump signal period (which can be much longer than the pump signal envelope period) is attributed to the presence of the Heaviside function in the stress–strain relationship.

It is expected that these results can help in the acoustic probing of the granular media (contact statistic distribution, average stress) and of the interfaces between solids (such as cracks).

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References