ACOUSTIC MODES IN DUCT WITH PARALLEL SHEAR FLOW AND VIBRATING WALLS

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Abstract

Numerical and theoretical investigation of the influence of subsonic shear flow on the axisymmetric modes in a circular duct with vibrating walls are reported. The considered geometry is an elastic duct filled with fluid in shear motion, and surrounded by vacuum. Then, we solve the coupled problem in order to find the eigenmodes and the dispersion diagrams. In this first study, the problem is considered under axisymmetric assumption in order to simplify calculations in spite of the fact that, in real problems, the first vibration and acoustic modes are not axisymmetric. In the solid, i.e. the wall, exact axisymmetric elasticity equations are used to yield analytical expressions. In the fluid, the classical Prandtl-Brown equation governing transverse duct modes with shear flow is used. The boundary condition at the interface fluid/solid, responsible for the coupling, is chosen to be continuity of stress and normal displacement. In the solid, an approximate theory, called thin shell or Donnel theory, is also tried. Results show the influence of shear flow on dispersion diagram of acoustic modes in the duct.

List of symbols

\( \rho \): density in the solid
\( \rho_0 \): density in the fluid
\( E \): Young modulus
\( \nu \): Poisson’s ratio
\( \lambda, \mu \): Lamé constants in the solid
\( J_n \): Bessel function of the first kind of order \( n \)
\( Y_n \): Bessel function of the second kind of order \( n \)
\( c \): fluid acoustic free wave speed
\( c_L \): longitudinal free wave speed in the solid
\( c_T \): transversal free wave speed in the solid
\( \omega \): circular frequency
\( \Omega \): reduced frequency
\( u_r, u_z \): displacement
\( p \): pressure
\( M(r) \): Mach number
\( M_0 \): mean Mach number
\( r_{rr}, r_{rz} \): stress component in the solid
\( a \): interior radius of the solid
\( b \): exterior radius of the solid
\( x \): \( a/b \), radius ratio
\( R \): mean radius
\( e \): shell thickness
\( \zeta \): longitudinal wavenumber
\( K \): reduced wavenumber, \( \zeta c/\omega \)

I. Introduction

Wave propagation in cylindrical shells with internal mean flow is subject of theoretical interest and of importance to many industrial applications (for instance propagation in ducts).

A first approach to this complex problem is to study the modes in the duct ignoring the propagation outside the duct; in this case, the exterior domain is assumed to be the vacuum. Then, one has to solve the coupled modal problem in the fluid filling the shell, and in the shell (the wall of the duct). Investigations of shear mean flow effects on duct modes have begun with the study of Prandtl-Brown.\(^1\) Since then, many authors have presented results in ducts with rigid lined wall.

On the other hand, the presence of vibrating wall has been the subject of articles centered on fluid-loading effects on modes of the cylindrical shell. Thin shell theory as well as exact theory have been applied.\(^2,3,4\) Some authors have taken into account both wall vibration and mean flow effect,
but, in these studies, the flow was assumed to be uniform. In this paper, our aim is to investigate the modes in duct with wall vibration and shear mean flow.

In section II, we present the governing equations in the fluid and in the solid, and the boundary condition between them. Then, in section III, some numerical results are presented, which are applications of the theory.

II. Governing equations

A. Equation in the fluid

When the axisymmetric acoustic pressure in a duct with parallel shear flow is written

\[ p(r_*, z_*, t) = P(r_*)e^{i(\omega t + \zeta z_*)}, \tag{1} \]

and ignoring visothermal effects and assuming adiabaticity, the equation governing the axisymmetric transverse modes in duct with shear flow was first given by Pridmore-Brown. In dimensionless form, it is

\[ \Delta_t P - \frac{2KM'}{1 + KM} \frac{dP}{dr} + \Omega^2((1 + KM)^2 - K^2)P = 0 \tag{2} \]

where \( \Delta_t = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) \) is the transverse laplacian, \( M(r) \) the local Mach number \( (M' = dM/dr) \), \( r = r_*/b \) the reduced frequency and \( K = \zeta c/\omega \) the reduced wavenumber.

In non-uniform shear flow, equation (2) must be solved numerically. The boundary condition will be given by the coupling with the cylindrical shell. For rigid walls, this condition is \( dP/dr = 0 \). For vibrating walls, the condition is derived in the next sections.

B. Equations in the solid

The exact equation governing the displacement \( \bar{u} \) in the solid is \( \rho \)

\[ \rho \frac{\partial^2 \bar{u}}{\partial t^2} = (\lambda + \mu) \nabla \cdot (\nabla \bar{u}) + \mu \Delta \bar{u} \tag{3} \]

which is valid for arbitrary wall thickness. When this equation is specialized to the axisymmetric case, it becomes

\[ \rho \frac{\partial^2 u_r}{\partial t^2} = (\lambda + 2\mu) \left( \frac{\partial^2 u_r}{\partial r_*^2} + \frac{1}{r_*} \frac{\partial u_r}{\partial r_*} - \frac{u_r}{r_*^2} \right) \]

\[ + (\lambda + \mu) \frac{\partial^2 u_z}{\partial r_* \partial z_*} + \mu \frac{\partial u_z}{\partial z_*^2}, \tag{4} \]

\[ \rho \frac{\partial^2 u_z}{\partial t^2} = (\lambda + \mu) \left( \frac{\partial^2 u_r}{\partial r_*^2} + \frac{1}{r_*} \frac{\partial u_r}{\partial r_*} \right) \]

\[ + (\lambda + 2\mu) \frac{\partial^2 u_z}{\partial z_*^2} + \mu \frac{\partial u_z}{\partial r_*} + \frac{1}{r_*} \frac{\partial u_z}{\partial r_*} \tag{5} \]

When the modes are sought in the form

\[ u_r(r_*, z_*, t) = f(r_*)e^{i(\omega t + \zeta z_*)}, \tag{6} \]

\[ u_z(r_*, z_*, t) = g(r_*)e^{i(\omega t + \zeta z_*)}, \tag{7} \]

an analytical solution of this problem is available, and in dimensionless form it can be written

\[ f = -\alpha(A_1 J_1(\alpha r)) + B_1 Y_1(\alpha r)) \]

\[ + j\Omega K(A_2 J_1(\beta r)) + B_2 Y_1(\beta r)) \]

\[ g = j\Omega K(A_3 J_0(\beta r) + B_3 Y_0(\beta r)) \]

\[ - \beta(A_2 J_0(\beta r) + B_2 Y_0(\beta r)), \tag{8} \]

where \( A_1, B_1, A_2 \) and \( B_2 \) are coefficients to be determined, \( \gamma_L = c/c_L, \gamma_T = c/c_T, \) \( \alpha^2 = \Omega^2(\gamma_L^2 - K^2) \) and \( \beta^2 = \Omega^2(\gamma_T^2 - K^2) \) with the longitudinal and transverse celerities

\[ c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \tag{9} \]

\[ c_T = \sqrt{\frac{\mu}{\rho}} \tag{10} \]

The stress tensor components are

\[ \tau_{rr} = \lambda \left( \frac{1}{r_*} \frac{\partial u_r}{\partial r_*} + \frac{\partial u_z}{\partial z_*} \right) + 2\mu \frac{\partial u_r}{\partial r_*}, \tag{11} \]

\[ \tau_{rz} = \mu \left( \frac{\partial u_z}{\partial r_*} + \frac{\partial u_r}{\partial z_*} \right). \tag{12} \]

So, by using solutions (7), it is possible to write

\[ \tau_{rr} = \frac{\lambda}{b} \{ N_{11} A_1 + N_{12} B_1 + N_{13} A_2 + N_{14} B_2 \} \tag{13} \]

with

\[ N_{11} = - \left[ \Omega^2 \gamma_L^2 + \frac{2\mu}{\lambda} \alpha^2 \right] J_0(\alpha r) + \frac{2\mu}{\lambda} \frac{J_1(\alpha r)}{r}, \tag{14} \]

\[ N_{12} = - \left[ \Omega^2 \gamma_T^2 + \frac{2\mu}{\lambda} \alpha^2 \right] Y_0(\alpha r) + \frac{2\mu}{\lambda} \frac{Y_1(\alpha r)}{r}, \tag{15} \]

\[ N_{13} = \frac{2\mu}{\lambda} j\Omega K \left( \beta J_0(\beta r) - \frac{J_1(\beta r)}{r} \right), \tag{16} \]

\[ N_{14} = \frac{2\mu}{\lambda} j\Omega K \left( \beta Y_0(\beta r) - \frac{Y_1(\beta r)}{r} \right), \tag{17} \]

and

\[ \tau_{rz} = \frac{\mu}{b} \{ N_{21} A_1 + N_{22} B_1 + N_{23} A_2 + N_{24} B_2 \} \tag{18} \]

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with
\[N_{21} = -2j\Omega K\alpha J_1(\alpha r),\]
\[N_{22} = -2j\Omega K\alpha J_1(\alpha r),\]
\[N_{23} = J_1(\beta r) (\beta^2 - \Omega^2 K^2),\]
\[N_{24} = Y_1(\beta r) (\beta^2 - \Omega^2 K^2).\]  
(15)

Once the analytical expressions are known in the shell, the remaining constants \(A_1, B_1, A_2, B_2\) are to be related to fluid quantities \(P\) and \(dP/dr\) at the wall by the boundary conditions.

C. Boundary conditions

At the fluid/solid interface, i.e. for \(r = 1 (r_0 = b)\), the classical perfect slip boundary conditions are the continuity of normal stress and of the normal displacement and the vanishing of tangential stress:
\[
\begin{align*}
\tau_{rr} &= -p, \\
\tau_{r\theta} &= 0, \\
p/r &= \rho_0\omega^2 u_r.
\end{align*}
\]
(16)

At the solid-vacuum interface, i.e. for \(r = x (r_0 = a)\), the stress continuity gives
\[
\begin{align*}
\tau_{rr} &= 0, \\
\tau_{r\theta} &= 0.
\end{align*}
\]
(17)

So, it is possible to write the boundary conditions, involving stress tensor components, in matricial form;
\[
N\vec{A} = -\frac{b}{\lambda}p\vec{e}_i
\]
(18)

with
\[
N = \begin{pmatrix}
N_{11}(1) & N_{12}(1) & N_{13}(1) & N_{14}(1) \\
N_{21}(1) & N_{22}(1) & N_{23}(1) & N_{24}(1) \\
N_{11}(x) & N_{12}(x) & N_{13}(x) & N_{14}(x) \\
N_{21}(x) & N_{22}(x) & N_{23}(x) & N_{24}(x)
\end{pmatrix},
\]
(19)

\[
\vec{A} = \begin{pmatrix}
A_1 \\
B_1 \\
A_2 \\
B_2
\end{pmatrix}
\]
(20)

and
\[
\vec{e}_i = \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\]
(21)

The continuity of displacement yields
\[
\frac{dP}{dr} = \rho_0\omega^2 \vec{C} \cdot \vec{A}
\]
(22)

where
\[
\vec{C} = \begin{pmatrix}
-\alpha J_1(\alpha r) \\
-\alpha J_1(\alpha r) \\
\beta^2 - \Omega^2 K^2 \\
\beta^2 - \Omega^2 K^2
\end{pmatrix}
\]
(23)

At this point, the boundary condition that the pressure has to satisfy at the wall could be written directly
\[
\frac{dP}{dr} = -\frac{\rho_0\omega^2}{\lambda}g^2 h P
\]
(24)

with
\[
h = (\vec{C} \cdot \vec{e}_i).
\]
(25)

But it appears that this relation introduces numerical singularities due to the inversion of the matrix \(N\). We have found preferable to write the boundary condition at the wall as follow:
\[
\det(N) \frac{dP}{dr} + \rho_0\omega^2 \Omega^2 \det(L) P = 0
\]
(26)

with the matrix \(L\) defined by
\[
L = \begin{pmatrix}
C_1 & C_2 & C_3 & C_4 \\
N_{21}(1) & N_{22}(1) & N_{23}(1) & N_{24}(1) \\
N_{11}(x) & N_{12}(x) & N_{13}(x) & N_{14}(x) \\
N_{21}(x) & N_{22}(x) & N_{23}(x) & N_{24}(x)
\end{pmatrix}
\]
(27)

where the \(C_i\) are the components of \(\vec{C}\).

Now, it is possible to integrate the Prandtl-Brown equation (sec II.A.), with the above boundary condition.

D. An approximation: thin shell theory

When the thickness of the shell is low enough, it is interesting to use an approximate theory valid for thin shell. One of these approximate formulations is the Donnel theory
\[
R^2 \left[ \frac{\partial^2}{\partial z^2} - \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} \right] u_z + \nu \frac{\partial u_z}{\partial z} = 0
\]
(28)

\[
\nu \frac{\partial u_z}{\partial z} + \left[ 1 + \delta^2 R^4 \frac{\partial^2}{\partial z^2} + \frac{R^2}{c_i^2} \frac{\partial^2}{\partial t^2} \right] u_r = \frac{R \frac{1}{c_i^2} \rho \omega^2 p}{\nu}
\]
(29)

where \(R\) is the mean radius of the shell, \(\varepsilon\) the shell thickness, \(\delta^2 = \varepsilon^2/12R^2\) and
\[
c_i = \sqrt{\frac{E}{\rho(1-\nu^2)}}.
\]
(30)
By taking a dependence of the form: $e^{i(\omega t + \zeta z)}$, these equations become, in dimensionless form

$$\Omega_S (\gamma_l^2 - K^2) U_r + j\nu K U_r = 0$$  \hspace{1cm} (31)

$$j\nu K \Omega_S U_z + (1 + \delta^2 \Omega_S^4 K^4 - \gamma_l^2 \Omega_S^2) U_r = \frac{R}{c' \rho e} \frac{1}{P}$$  \hspace{1cm} (32)

where $\gamma_l = c/q$, $\Omega_S = \omega R/c$ and $U_r, U_z = u_r, u_z / R$.

On the other hand, continuity of displacement imposes that

$$\frac{dP}{dr} = \rho \omega^2 R U_r,$$  \hspace{1cm} (33)

hence the problem can be formalized as the following eigenproblem

$$
\begin{pmatrix}
\Omega_S (\gamma_l^2 - K^2) & j\nu K & j\nu K \\
0 & (1 + \delta^2 \Omega_S^4 K^4 - \gamma_l^2 \Omega_S^2) & 0 \\
0 & \rho \omega^2 R & -\frac{R}{c' \rho e}
\end{pmatrix}
\begin{pmatrix}
U_r \\
U_z \\
1
\end{pmatrix} = 0
$$

or likewise

$$-\frac{dP}{dr} \left[ (\gamma_l^2 - K^2) \left(1 + \delta^2 \Omega_S^4 K^4 - \gamma_l^2 \Omega_S^2 \right) \right] + (\nu K)^2 \left[ \frac{\rho \omega^2 R}{\rho e} \Omega_S^2 (\gamma_l^2 - K^2) P \right] = 0$$

This is the boundary condition that the pressure has to satisfy at the wall, during the integration of the Prandtl-Brown equation, in the thin shell approximation.

**III. Results**

In this section, the theory developed in section II is applied to some configurations by numerical integration. For all cases, the following parameters are the same: $\rho = 7910\text{kg/m}^3$, $\rho_0 = 1000\text{kg/m}^3$, $c = 1500\text{m/s}$, $\gamma_L = 0.26$, $\gamma_T = 0.48$ and $\gamma = 0.28$.

In Figures 1 to 4, solutions of the dispersion diagram given by exact and approximate (thin shell) theories are compared without mean flow. The agreement is better for thinner shell (Figures 1 and 2) with $x = 1.04$ and we can see that the approximate formulation is good up to the ring frequency. Complementary comparisons had been made already by Greenspon.\textsuperscript{11,12}

The effect of shear flow can be seen in Figure 5 for a parabolic profile. It is more pronounced on fluid dominated modes than on solid dominated ones. The role of the shape of the profile is displayed.

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Figure 3: No fluid, vacuum/solid/vacuum geometry, $K$ function of $\Omega$, $x = 1.2$, solid line: exact theory, o: thin shell theory.

Figure 5: Dispersion diagram, $K$ function of $\Omega$, parabolic mean flow profile, solid: $M_0 = 0$, dotted: $M_0 = 0.05$, dashed: $M_0 = 0.1$

Figure 4: fluid/solid/vacuum geometry $K$ function of $\Omega$, $x = 1.2$, solid line: exact theory, o: thin shell theory.

Figure 6: $K$ function of $M_0$ for mode 0 (upper branch), $\Omega = 4$, solid: parabolic profile, dashed: tenth power profile
in Figures 6 to 8 for the first two fluid dominated modes. One of the profile is parabolic, $M(r) = 2M_0(1 - r^2)$, and the other is a tenth power one $M(r) = 1.2M_0(1 - r^{10})$. No universal tendencies are visible. For instance the slope at the origin of the curve $K$ function of Mach number is dependent of the frequency in contrary to the case without wall vibrations.\(^\text{13}\)

Figure 9 shows the effect of mean shear flow on the dispersion diagram computed with the thin shell theory. Once again it appears that the fluid dominated modes are more sensitive to mean flow than the solid dominated modes.

**IV. Conclusion**

This work takes into account the wall elasticity for the calculation of the modes in shear flow ducts. The usual perfect wall boundary condition is replaced by a mixed boundary condition whose coefficients are determined by analytical solutions of elasticity in the wall. Both exact and approximate theories are presented for the shell vibration.

Our method enables to get wavenumber dependence on the Mach number and flow profile after numerical integration of the Prigmore-Brown equation. The inclusion of wall vibration appears to complexify the behavior of the dispersion properties in presence of shear flow. For instance, even for the mode 0, and for small Mach number $M_0$, the correction of the wavenumber due to $M_0$ is
dependent of frequency.

Further studies should include radiation outside the duct to model leaky modes.

References


