Acoustic Scattering in Duct With a Chaotic Cavity

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Summary
This work is dedicated to the experimental analysis of the statistics of the scattering of sound by a complex cavity in a duct. It is shown that the application of the Random Coupling Method based on the Random Matrix Theory can explain very well the fluctuations of reflection due to those cavities. An emphasis is made in this study to the effect of dissipation and of flow on the statistics of the scattering coefficients of the cavity.

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1. Introduction

In the airflow system of many industrial applications, large enclosures connected to ducts or complex duct networks can be found. In the aeronautical industry, a typical example is the air conditioning distribution system of an aircraft. Even for ducts where only the plane mode can propagate, the reflection and transmission coefficients from these enclosures can have very complex behaviors. Indeed, from the acoustical point of view, these cavities have a size much larger than the wavelength and an irregular geometry. Consequently, they create many resonances coupled to the duct, leading to strongly fluctuating scattering properties. In this case, due to huge complexity of the results, it is of great interest to get information about the statistics of the scattering. We can notice that, for the study of coupled structural vibrations, the statistical approach has been widely used, starting form the ideas of Statistical Energy Analysis [1, 2, 3, 4, 5, 6].

In the limit where the wavelength is small compared to the characteristic size of the scattering enclosure, wave propagation inside the enclosure can be modeled using ray trajectories. If the geometry is irregular and in the limit of short wavelengths, those wave systems can exhibit ray chaotic properties. This wave chaos has been intensively studied in the context of quantum mechanical waves [7, 8], however, the concepts that have been developed can generally be extended to classical waves such as acoustic [9, 10, 11, 12].

In particular, Random Matrix Theory (RMT) was shown to be an efficient tool to obtain results on the distribution of transmission and reflection coefficients [13, 14, 15, 16]. The idea behind the Random Matrix Theory of scattering is to assume that the dynamics of the wave is so irregular that it implies random distribution of the scattering coefficients. Then it remains to define the probability density of the scattering matrix; it is obtained by keeping only the fundamental properties of the propagation, i.e. the conservation of energy and/or the reciprocity. For instance, by keeping only the conservation of energy, the scattering matrix has to be unitary and the probability density is chosen as the uniform distribution on the compact set of unitary matrices [17].

In addition, two important effects have to be taken into account: i) the losses and ii) the imperfect coupling of the cavity with the waveguide. The introduction of losses is possible by suitable modeling of the attenuation [18, 19, 15, 16]. The imperfect coupling is often modeled by some averaging on frequencies or on different realizations of the experiment (see e.g. [15]). The Random Coupling Method (RCM) [20, 21, 22, 23] has been proposed as an alternative approach to tackle the problem of imperfect coupling and losses. It is based on the impedance matrix of the scattering cavity that can be decomposed in two parts: the so-called radiation impedance is associated with the scattering of the wave at the junction between the waveguide and the cavity, and the other part of the impedance is universal and associated with the random field in the cavity.

Of course, acousticians rarely encounter truly lossless systems in practice. Furthermore, in duct systems, the coupling between the duct and the cavity is far from perfect (reflection of the sound in a duct radiating in free space is not weak) and the flow is very often present. Thus in this paper, we explore experimentally a generic cavity in which we control both the flow and the dissipation and we compare these experimental results with results predicted by RCM. The plan of this paper is as follows. The principles of the RCM are reminded in sections 2 to 4. Then, in section 5, we describe the experimental apparatus. The experimental results are presented in section 6 to 8 for an empty cavity, a cavity with increased losses due to the presence of porous patches, and a cavity with slow flow.

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2. Scattering by a large cavity

The acoustical effect of a large cavity on a pipe system in the harmonic regime (convention $e^{i\omega t}$), see Figure 1, can be fully described by the scattering matrix $S$ that links the incoming waves on the cavity to the outgoing waves. Equivalently, the cavity can also be described with the impedance matrix $Z$ that links the acoustical velocity in each duct, at the junction duct-cavity, to the pressure in each ducts. In the case of the 2-port, as in Figure 1, this relation is written

$$
\begin{pmatrix}
    p_1 \\
    p_2
\end{pmatrix} = Z
\begin{pmatrix}
    u_1 \\
    u_2
\end{pmatrix},
$$

with

$$
Z = \begin{bmatrix}
    Z_{11} & Z_{12} \\
    Z_{21} & Z_{22}
\end{bmatrix}.
$$

(1)

In the case of a 1-port ($u_2 = 0$), Equation (1) reduces to $p_1 = Z_{11}u_1$. In the following, the pressure $p$ and the velocity $u$ are respectively normalized by $\rho_0c_0^2$ and $c_0$ where $\rho_0$ is the density and $c_0$ is the sound velocity. With this normalisation, the impedance is normalized by $\rho_0c_0$. The relation between the scattering and impedance matrices is

$$
S = (Z + I)^{-1}(Z - I),
$$

(2)

where $I$ is the identity matrix. The choice of using the impedance matrix, rather than the more classical scattering matrix comes from the use in the following of the Random Coupling Model (RCM) [20, 21, 22, 23]. It will allow us to decompose the scattering processes. Indeed, the global behavior of the cavity can be separated in two contributions: one is linked to radiation of waves from duct into the cavity and the second is linked to the intrinsic behavior of the cavity. The global impedance matrix can be written [23]

$$
Z = j\Imag(Z_{\text{rad}}) + \Re(Z_{\text{rad}})^{1/2} \Re(Z_{\text{rad}})^{1/2},
$$

(3)

where $\Re$ and $\Imag$ refer to the real and imaginary part, $Z_{\text{rad}}$ is a diagonal matrix whose elements are the complex radiation impedances of each port and $Y$ is matrix describing the intrinsic cavity behavior. Equation (3) is a key relation in the RCM: it allows us to decouple simply the impedance $Z$ into two parts. $Z_{\text{rad}}$ is the radiation impedance that is specific to the coupling between the cavity and the ducts. $Y$ is universal, representing the chaotic wave scattering, and it depends only on a loss $\alpha$ that will appear later. The way to obtain $Z_{\text{rad}}$ and $Y$ is described in the two following sections. In the case of a 1-port, this equation reduces to

$$
Z_{11} = j\Imag(Z_{\text{rad}}) + \Re(Z_{\text{rad}}) \zeta,
$$

(4)

where $\zeta$ is the (1,1) element of $Y$.

3. Radiation impedance

The radiation impedance $Z_{\text{rad}}$ of each port is the impedance of the duct for the same geometry of the duct-cavity junction but with the side walls of the cavity removed to infinity so that no waves escaping from one duct can return into it [20]. The real part of the radiation impedance contains information about the waves carried away from the duct while the imaginary part contains information about the near field of the duct into the cavity.

At very low frequencies ($k_0a \ll 1$ where $a$ is the inner radius of the duct, $k_0 = \omega/c_0$ is the wavenumber, $\omega = 2\pi f$, $f$ is the frequency, $c_0$ is the sound velocity), the radiation impedance is very weak and the reflection coefficient $R$ is close to -1, similarly to the radiation from a duct to free space. It means that the intrinsic behavior of the cavity has a very small influence on $R$. As the frequency increases, more and more acoustic energy enter into the cavity and it becomes important to take the behavior of the cavity into account. In the case of a duct emerging into a large cavity, a 1-port system, $Z_{\text{rad}}$ can be approximated by the Rayleigh radiation impedance of a piston mounted in an infinite baffle [24]:

$$
Z_{\text{rad}} = 1 - \frac{J_1(2k_0a)}{k_0a} + j \frac{H_1(2k_0a)}{k_0a},
$$

(5)

where $J_1$ is the order 1 Bessel function of the first kind and $H_1$ is the order 1 Struve function.

4. Universal statistics of the chaotic cavity

When the wavelength of the acoustical waves becomes much smaller than the size of the cavity, the number of modes becomes very large. Due to the lack of regularity of the boundaries of the cavity, the propagation in the cavity is randomized and it can be described with the wave chaos theory. An interesting aspect of wave chaotic systems is that, despite their apparent complexity, they all possess certain universal statistical properties in their scattering fluctuation characteristics. These statistical fluctuations are well-described by the statistical properties of ensembles of large random matrices. One distinguishes in particular the Gaussian unitary ensemble (GUE) which is associated with wave operators with broken time-reversal symmetry and the Gaussian orthogonal ensemble (GOE) for systems which are time-reversal invariant. For chaotic cavities, the intrinsic cavity scattering matrix $S$, linked to the matrix $Y$ of Equation (5), by $S_{\text{RMT}} = (Y - i\lambda)(Y + i\lambda)^{-1}$ can be computed by [25]

$$
S_{\text{RMT}} = U \begin{bmatrix}
    \sqrt{1 - T_1} & 0 \\
    0 & \sqrt{1 - T_2}
\end{bmatrix} U^T
$$

(6)
where the matrix \( U \) is uniformly distributed in the unitary group, \( U^T \) is the transpose of \( U \). The coefficients \( T_1 \) and \( T_2 \) in the diagonal matrix gives the absorption strength. Their joint statistical distribution can be computed for systems which are time-reversal invariant by [19]

\[
P(T_1, T_2) = \frac{1}{8} T_1^{-4} T_2^{-4} \exp \left( -\frac{\gamma}{2} \left( \frac{1}{T_1} + \frac{1}{T_2} \right) \right)
\]

\[
\cdot \left| T_1 - T_2 \right| \left[ \gamma^2 (2 - 2e^{-\gamma} + \gamma + \gamma e^{-\gamma}) \right.
\]

\[
- \gamma (T_1 + T_2) (6 - 6e^{-\gamma} + 4\gamma + 2\gamma e^{-\gamma} + \gamma^2)
\]

\[
+ T_1 T_2 (24 - 24e^{-\gamma} + 18\gamma + 6\gamma e^{-\gamma} + 6\gamma^2 + \gamma^3) \right].
\]

where \( \gamma = 4\pi a \) and \( a \) is the loss parameter. The distribution in case of broken time-reversal symmetry is much more lengthy and it can be found in [19]. From the relation \( Y = (I - S_{\text{RMT}})^{-1}(I + S_{\text{RMT}}) \), and from the distribution of \( S_{\text{RMT}} \), the statistical distribution of the impedance is obtained by computing the matrix \( Y \) many times in order to give a sufficiently large set of samples of \( Z \). The statistical description of the cavity is thus generated only by giving the loss parameter \( a \) and by knowing if the system is time-reversal invariant or not.

In the following, the measurements in the 1-port case are compared to the results of the RCM.

### 5. Experimental setup

In order to experimentally determine the reflection coefficient of a duct with an attached cavity (see Figure 2), the forward and backward pressure propagating waves need be measured in the duct. In the relevant frequency range (0–4000 Hz), only plane wave can propagate in the duct (radius 15 mm) and the pressure is written

\[
p(x, \omega) = p^+ e^{-j\beta x} + p^- e^{j\beta x}
\]

the x-axis origin has been set at the junction between the duct and the cavity. The reflection coefficient \( R \) at the origin \( (x = 0) \) is linked to the transfer function \( H_{21} \) between two microphones 1 and 2 by [26]

\[
R = \frac{p^-}{p^+} = \frac{H_{21} e^{-j\beta x_1} - e^{-j\beta x_2}}{e^{j\beta x_1} - H_{21} e^{j\beta x_2}}
\]

and the normalized entrance impedance is given by \( Z = (1 + R)/(1 - R) \).

The duct consists of a straight steel pipe with an inner radius equal to \( a = 15 \) mm and a thickness of 4 mm. Microphones B&K 4938 with Nexus 2690 amplifiers measure the acoustic pressures. A first set of measurements was made from 300–4000 Hz. To cover all the frequency range, five microphones were used. The distances between successive microphones are respectively: 30, 100, 375 and 597 mm in order to optimize the identification of propagating pressure waves. In the second set of measurements the frequency range is reduced to 3000–3500 Hz. In this case only two microphones are needed and the distance between those two microphones is 30 mm. To insure that the evanescent modes, caused by the discontinuity between the duct and the cavity, are negligible, the first microphone is located at a distance of 435 mm from the junction. The relative calibration of the microphones has been made as described in [27]. A high frequency resolution (frequency steps of 0.5 Hz) was used during the sine sweep measurement to distinguish the different cavity modes.

The cavity is a parallelepiped box of 0.924 \( \times \) 0.962 \( \times \) 0.565 m\(^3\). The box is made out of wood plates (thickness 20 mm) with glued stiffeners on the external faces to avoid, as much as possible, sound losses through the box enclosure. A rigid cylinder (diameter 0.2 m), half a cylinder, and two quarter of cylinder have been added into the box to break all its symmetries (See Figure 2) and to insure a chaotic behavior of rays in this box. The position of the vertical cylinder is changed to ensure various realizations of the chaotic acoustical field. The mean-free path of the sound rays (average distance between successive reflections) inside the cavity can be estimated by \( 4V/S = 0.44 \) m where \( V \) is the volume of the cavity and \( S \) is the total inside area of cavity.

Up to three panels of porous material can be inserted in the box to control the losses. This porous material is a melamine foam (panels of 300 mm \( \times \) 200 mm \( \times \) 50 mm). The panels are maintained by distortable rods that allow a variable orientation of the panels, see Figure 2b.

A flow can be produced inside this CB by an internal ventilation fan, see Figure 2c. This fan can provide a flow rate of 350 m\(^3\)/h and has a diameter of 150 mm. The velocity at the fan exit is 5.5 m/s (1.6 % of the sound velocity). The mean velocity field in the CB is complex and has not been measured.
6. Results for an empty box

The absolute value of the reflection coefficient $R$ measured for the CB is depicted in Figure 3. This figure shows that the overall trend of reflection coefficient is decreasing as a function of the frequency. Superimposed to this trend, there are rapid fluctuations. As describe by the relation (4), the entrance impedance of the cavity can be separated in two contributions: one is linked to radiation impedance $Z_{rad}$ and relates to the slowly varying part, the second is linked to the chaotic behavior of the cavity and relates to the fluctuations. The reflection coefficient $(1 - Z_{rad})/(1 + Z_{rad})$ where $Z_{rad}$ is obtained from Equation (5) is plotted on Figure 3 (Red thick continuous line) and gives a good description of the average behavior of $R$.

The radiation impedance $Z_{rad}$ takes into account the imperfect coupling between the field within the chaotic cavity and the incoming and outgoing waves in the connected duct. By suitably accounting for these coupling details, only the universal scattering properties of the chaotic cavity should remain. However, there may exist acoustical rays that leave the tube and soon return to it. Those short ray trajectories lead to non-universal contributions. Yeh et al. [22] propose to take into account this effect by a frequency sliding average of data. This frequency smoothing gives an average impedance $Z_{avg}$. This impedance suppresses the impedance fluctuations caused by long trajectories and reveals the features associated with short trajectories. The smoothing is made by convoluting the experimental results with a Hanning window of bandwidth $\Delta f$ of 60 Hz. In this case, $Z_{avg}$ takes into account the trajectories with a path length smaller than $c_0/\Delta f \approx 6$ m. The normalized universal impedance $\zeta$ can be found from the impedance of the cavity by $Z = |\mathcal{R}(Z_{avg}) + i\mathcal{I}(Z_{avg})| \zeta$. The real part of $Z$, $Z_{avg}$ and $Z_{rad}$ are plotted in Figure 4a and the real part of $\zeta$ in Figure 4b.

The probability density function (PDF) of the universal impedance $\zeta$ for the frequency band 3000–3500 Hz is

\[
\zeta \approx 1/(\pi \sigma^2).
\]

Computing the variance on a sliding window of 500 Hz, the variation of the loss parameter as a function of the frequency can be computed and is shown on Figure 5. It can be see that the value of the variance of the real and of the imaginary part of $\zeta$ are equal, in agreement to the theory. The loss parameter $\alpha$ increases with frequency and the other statistical properties had to be computed on a windows where $\alpha$ is almost constant and that contains enough points to have a good convergence of the statistics. In the following, a window of 500 Hz with 1000 samples is chosen and 3 realizations are done by moving the cylinder, thus the statistics are made with 3000 points.

The probability density function (PDF) of the universal impedance $\zeta$ for the frequency band 3000–3500 Hz is
given in Figure 6 for the empty box. On this figure, the experimental results are compared with the RMT results with a loss parameter $\alpha = 4.15$ which is the average of the loss parameter deduced from the variance over the investigated frequency band (see Figure 5). The experimental results are in very good agreement with the RMT results. The value of the loss parameter indicates that there is some losses in the empty box even if some efforts have been made to minimize them. The reverberation time (the time for a decreasing of the field by 60 dB) of this box is 0.74 s. The main dissipation process is supposed to be the coupling of the interior acoustical field with small vibrations of the walls.

7. Effect of absorbing porous material

To test the effect of larger absorption, porous panels are introduced in the box. It can be seen in Figure 2b that those panels are put inside the volume. The main expected effect is to change the averaged wave number for the propagation inside the box by introducing a complex part in it. The boundary conditions on the wall of the box are supposed to be unchanged.

The effects on the reflection coefficient of adding some absorption can be seen in Figure 3: The amplitude of the fluctuating part is decreasing when absorption is added. With higher absorption, the resonances are smoother and the number of visible resonance decreases. This implies that the variance of the fluctuating part decreases.

The variation of the loss parameter as a function of the frequency for the 3 cases with porous panels can be seen on Figure 7. The loss parameter $\alpha$ increases with the number of porous panels and with frequency.

On Figure 8, the experimental values of the PDF of $\zeta$ is plotted for the 3 cases with porous panels. The experimental values (symbols) are compared to the RMT values (lines) with $\alpha = 4.15, 15.3, 31.8$ and 70.7. The agreement between experimental and RMT results is good. It demonstrates that all the statistics of the CB can be computed from an unique parameter: the loss parameter that can be computed from the variance of the universal impedance $\zeta$.

8. Effect of flow

A ventilation fan is introduced in the CB to induce a circulation of the air in the box. The main effect of the mean flow in the box is to break the time reversal symmetry [14]. In this case, the statistics of the impedance without time reversal symmetry is changed compared to the impedance with time reversal symmetry. The same method as before is used by changing only the universal $S_{\text{RMT}}$ distribution (from time reversal to non time reversal symmetry, see [19]). This universal statistical distribution still depends the unique loss parameter $\alpha$. 

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**Figure 5.** (Colour online) Loss parameter $\alpha$ as a function of the frequency obtaining by computing the variance of the real (red circle) and the imaginary part (blue cross) of the universal impedance $\zeta$. The variance is computed on a sliding window of 500 Hz.

**Figure 6.** Probability density function of $\zeta$ for CB on the frequency band 3000–3500 Hz. (a) Real part of $\zeta$, (b) Imaginary part of $\zeta$. Circles: measurements, line: RMT results.

**Figure 7.** Loss parameter $\alpha$ as a function of the frequency for the empty box (circles) and for the box with panels of porous material (crosses: 1 panel, diamonds: 2 panels, squares: 3 panels).
9. Conclusion

Experiments have been conducted to measure the acoustical reflection coefficient of a box which can exhibit ray chaotic properties. We have verified that the normalized cavity impedance describes universal properties of the reflection fluctuations by properly taking into account the coupling process between the measurement duct and the cavity. The improvement of normalized impedance can be obtained by a mixing of ensemble averages on various realizations and of frequency averages over bandwidth containing several oscillations. The results show that the direct processes (short ray trajectories) have to be removed to obtain converged statistics. The Random Coupling Method (RCM) provides a good description of the one-port wave chaotic system. The reflection fluctuation statistics depend only on a single control parameter characterizing the cavity loss in the realm of intermediate to high losses induced by porous material inside the box. The breaking of the time
reversal symmetry by a small airflow inside the box is also successfully described by the RCM.

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References