Scattering by Finite Periodic $\mathcal{PT}$-Symmetric Structures

V. Achilleos, Y. Aurégan, and V. Pagneux

Laboratoire d’Acoustique de l’Université du Maine, UMR CNRS 6613 Av. O. Messiaen, F-72085 LE MANS Cedex 9, France

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In this work, we study the transmission properties of one-dimensional finite periodic systems with $\mathcal{PT}$ symmetry. A simple closed-form expression is obtained for the total transmittance from a lattice of $N$ cells, that allows us to describe the transmission minima (maxima) when the system is in the $\mathcal{PT}$-unbroken (broken) phase. Utilizing this expression, we provide the necessary conditions, independent of the number of cells, for the occurrence of a coherent perfect absorber and laser for any finite $\mathcal{PT}$-symmetric periodic potential. Under these conditions, we provide a recipe for building finite periodic structures with near perfect absorption and extremely large amplification.

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Studying the interplay between losses and gain in wave propagation has recently attracted considerable attention, stimulated by the discovery of $\mathcal{PT}$-symmetric [1] systems. Theoretical efforts where initially focused on extending Hermitian quantum theories [2,3] using non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians with real spectra. It has been, however, realized that the notion of $\mathcal{PT}$ symmetry and the corresponding phenomena can be readily extended to other physical systems. Experiments emulating $\mathcal{PT}$-symmetric Hamiltonians are performed in diverse physical settings including optical waveguides [4,5], microring resonators [6–10], audible acoustics [11,12], optomechanics [13], spin waves [14], or atomic systems [15]. The interest in such systems is motivated by the extraordinary wave properties, especially around exceptional points [16,17], which are otherwise unattainable in Hermitian systems including unidirectional propagation [18–21], enhanced sensitivity [22], and coherent perfect absorbers and lasers (CPA lasers) [23,24]. The CPA laser, especially in optics, is of great technological interest since it may lead to multifunctional devices, acting on the same time as absorbers, lasers, or modulators. Experimental observation of the CPA laser was first realized in an electronic circuit analog [25] and has been recently reported in an optical setting [26].

$\mathcal{PT}$ symmetry requires a delicate exact balance between gain and loss, which makes the experimental observation of the $\mathcal{PT}$-phase transition as well as the CPA laser a very challenging task. One way to bypass the difficulties is to build elemental two-component systems and fine-tune loss or gain the obtain exact $\mathcal{PT}$ symmetry. On the other hand, periodic structures with $\mathcal{PT}$ symmetry are of special interest [27–31], and have been shown to have interesting properties including unusual band structure and Bloch oscillations [32–37]. They offer a unique opportunity for generating novel devices, and constitute a promising setting to overcome the consequences of losses in many applications including the growing field of metamaterials.

Although many studies exist for the case of infinite periodic $\mathcal{PT}$-symmetric systems, the scattering in finite periodic systems composed of $N$ number of cells has been less investigated. In many cases (for example, Ref. [18] and the experimental works [19,26,38]) finite systems are studied using a coupled mode theory around the Bragg points and an approximate transfer matrix describes the relevant phenomena.

It was recently shown in Ref. [39] that asymmetric transmission resonances (ATR) [40] and the CPA-laser points of a finite number $\mathcal{PT}$-symmetric dielectric layers highly depend on the number of cells. A finite lattice of $N$-quantum $\mathcal{PT}$-symmetric scatterers was studied in Ref. [41], focusing on the difference between parallel and in-series coupling of the scatterers, but also relating the singular points of the scattering matrix, the unit cell, and the $N$-cell structure. More recently, the singular value spectrum of a finite periodic electromagnetic structure was studied in Ref. [42], focusing on the CPA-laser point.

The purpose of this Letter is to give simple closed-form expressions, describing the transmission and the CPA-laser points, from an arbitrary one-dimensional finite periodic $\mathcal{PT}$-symmetric scatterer. The expression for the total transmission from the finite system is given in Eq. (7), which depends on the unit cell transmission, the Bloch phase, and the total number of cells. An envelope function given in Eq. (9) captures the minima of transmission in the $\mathcal{PT}$-unbroken phase or the maxima in the $\mathcal{PT}$-broken phase. Using this simple function, the necessary, $N$ independent, conditions for a CPA laser are found for any finite periodic $\mathcal{PT}$-symmetric potential in one dimension. These conditions are given in Eq. (10). Finally, we show that when the necessary requirements obtained by the envelope function are met, near perfect absorption and extremely strong amplification at the same frequency can be obtained even away from the CPA laser.

In the following, we study one-dimensional scattering systems satisfying the stationary Schrödinger equation.

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\[ \psi'' + [k^2 - V(x)]\psi = 0, \]  

(1)

or the Helmholtz equation relevant to optical Bragg gratings,

\[ \psi'' + k^2 n^2(x)\psi = 0, \]  

(2)

where \( \psi(x) \) is the wave field and primes denote derivatives with respect to \( x \). The \( PT \) symmetry of the potential is established when \( V(x) = V^*(x) \), where the star denotes complex conjugation, while the normalized refractive index \( n(x) \) also is \( PT \) symmetric when \( n(x) = n^*(x) \). We study space periodic potentials of period \( l \) satisfying \( V(x + l) = V(x) \) and \( n(x + l) = n(x) \) [see Fig. 1(a)]. For both equations, the scattering matrix for the unit cell with length \( l \) has the form

\[ S_1 = \begin{pmatrix} r_L & t \\ t & r_R \end{pmatrix}, \]  

(3)

where \( t, r_L, \) and \( r_R \) are the transmission, reflection from left and right coefficients, respectively. The corresponding transfer matrix defined in the region \(-l/2 \leq x \leq l/2\), is \( PT \) symmetric and is given by

\[ M_1 = \begin{pmatrix} 1/|t| & r_R/|t| \\ -r_L/|t| & 1/|t| \end{pmatrix}. \]  

(4)

\( M_1 \) has a unitary determinant, leading to the following relation \( r_L r_R = t^2 (1 - T^{-1}_1) \) and, subsequently, to the following “conservation law” \( |T_1 - 1| = \sqrt{R^{(1)}_L R^{(1)}_R} \). \( T_1 = |t|^2 \) is the total transmittance from the unit cell and \( R^{(1)}_{LR} = |r_L r_R|^2 \) the total reflectances of the unit cell from left and right. Since the system does not conserve time reversal symmetry \( R^{(1)}_L \neq R^{(1)}_R \).

Depending on the parameter values of the unit cell, the transmittance may either be \( T_1 < 1 \) or \( T_1 > 1 \). The eigenvalues of \( M_1 \) are written as \( \lambda_1 = 1/\lambda_2 = \exp[i\phi] \) resulting in

\[ \cos \phi = \text{Re}(1/t). \]  

(5)

For an infinite periodic potential, \( \phi \) corresponds to the Bloch phase.

Transmittance.—We now focus on the scattering from a periodic structure composed by \( N \) cells. Both the potential \( V(x) \) or the refractive index \( n(x) \) are \( PT \) symmetric in a region \(-L/2 < x < L/2\), where the total length of the scatterer is \( L = Nl \). Using the Chebyshev identity we can write the transfer matrix for \( N \) cells [39,42], which has the form [43]

\[ M_N = \begin{pmatrix} 1/|t| & \sin(N\phi) \\ -r_L/|t| & \sin(N\phi) \end{pmatrix}. \]  

(6)

Since the potential is \( PT \) symmetric, the conservation relation is generalized to \( |T_N - 1| = \sqrt{R^{(N)}_L R^{(N)}_R} \). Using the fact that \( M_N \) also has a unitary determinant, and that it is \( PT \) symmetric, from Eq. (6) we obtain the total transmittance from \( N \) cells as

\[ \frac{1}{T_N} = 1 + \left( \frac{1}{T_1} - 1 \right) \sin^2(N\phi). \]  

(7)

The transmission is obtained as a function of \( \phi \), \( T_1 \), and \( N \). Interestingly enough, Eq. (7) has the same form as the transmission through a periodic scatterer without loss or gain [45,46]. In Refs. [45,46] the expression of Eq. (7) is derived for the conservative system exploiting the unitarity of \( S_1 \). On the contrary, here we use the “generalized unitarity relation” (see Ref. [40]). To our knowledge this simple and useful expression has not been established in the literature for \( PT \)-symmetric periodic potentials. Note also that, Eq. (7) is not valid for a system featuring either only loss or only gain, (see the Supplemental Material, Sec. 2 [43]).

The simplicity of Eq. (7) allows us to identify most of the characteristics of the scattering problem with simple arguments, and as we present below, it permits us to extract the necessary conditions for the CPA laser. To illustrate the use of Eq. (7) we study a \( PT \)-symmetric periodic potential composed by \( \delta \) barriers for Eq. (1). The unit cell potential is given by

\[ V(x) = \lambda \delta(x) - i \gamma \left( \delta \left( x - \frac{l}{4} \right) - \delta \left( x + \frac{l}{4} \right) \right). \]  

(8)

defined in the region \(-l/2 < x < l/2\) with a total length \( l \).

In the unbroken phase \( (T_1 < 1) \), the \( PT \)-symmetric potential has a similar behavior as in the conservative
case, and in order to be self contained we review these properties [45]. The transmission $T_N$ acquires $N - 1$ transmission resonances (TRs) within each propagating band, corresponding to the values of $N\phi = n\pi$ with $n = 1, 2, \ldots, N - 1$. An example of the transmittance from $N = 6$ cells in the $\mathcal{PT}$-unbroken phase, is shown in Fig. 1(d), featuring 5 TRs in each band. Note that, according to Eq. (7), additional resonances ($T_1 = 1$) of unit cell have to be added to the $N - 1$ stemming from the periodicity. Importantly, the reasoning for counting the TRs is based on the fact that the Bloch phase $\phi$ inside each transmitting band is a *monotonous* function describing the region $\phi \in [0, \pi]$, for the unbroken phase. The Bloch phase of the unit cell (8) with $\gamma = 1$, is shown in Fig. 1(b).

Additionally, the minima of transmission inside each band appear when $N\phi = n\pi/2$. Note that, as for any finite one-dimensional scattering problem, there is no limit of infinite $N$ that recovers the Bloch waves solution. Thus, when $N$ goes to infinity, $T_N$ has no limit for a given frequency.

In the broken phase ($T_1 > 1$), according to Eq. (7), the transmission changes drastically. Although there exist TRs at $N\phi = n\pi$, superradiant transmission peaks with $T_N > 1$ also appear at $N\phi = n\pi/2$. For values of $k$ between the beginning of the first band and $k = 2\pi$, which is now a region of propagation, $\phi$ is no longer *monotonic*. In fact, it starts from $\phi = 0$ and returns to 0 without passing from $\phi = \pi$ as shown in Fig. 1(c). An important consequence of the trajectory of $\phi$ is that the number of TRs is now *system dependent*.

**Envelope transmission function and CPA-laser condition.**—A simple envelope function [45], *independent* of the number of cells $N$, can be obtained by considering the points $\sin(N\phi) = 1$ which, using Eq. (7), leads to

$$\frac{1}{T_{env}} = 1 - \frac{1}{T_1} \sin^2 \phi. \quad (9)$$

In the unbroken phase ($T_1 < 1$), $T_{env}$ describes the minima of the total transmission $T_N$ [45] in the propagating bands, as in a conservative system. This is shown in Fig. 1(d) by the dashed black line. Even for a small lattice of six cells, this $N$ independent function captures the envelope of the minima efficiently. On the other hand, in the broken phase ($T_1 > 1$), $T_{env}$ describes the maxima of $T_N$ as shown in Fig. 1(e). Importantly, the envelope function $T_{env}$ becomes infinite in the region where $T_1 > 1$ when, according to Eq. (9), $\sin^2 \phi = (1 - 1/T_1)$ [see Fig. 1(e)]. Such an infinite transmission in a $\mathcal{PT}$-symmetric potential corresponds to a CPA-laser point [23,24]. The CPA laser appears when one eigenvalue of the total scattering matrix goes to infinity (laser) while the other vanishes (absorber). Because of this property, when the system is found to exhibit huge transmission it also expected to act as a near perfect absorber at the same frequency and for the same parameters.

Using Eq. (5), we find the necessary condition, which is $N$ independent, in order for a finite periodic system to feature a CPA laser

$$|t| > 1, \quad \text{and} \quad \text{Im}(t) = 0. \quad (10)$$

The condition of Eq. (10) depends solely on $t$ and is necessary for obtaining a CPA laser in the periodic structure. In Fig. 2(a), we show the single cell transmission $T_1$ (solid line) corresponding to the same values as in Fig. 1(e), around the superradiant region. The dashed line depicts $T_{env}$, which diverges at the value of the point where the imaginary part of $t$ [see Fig. 2(b)] crosses zero. For any finite system, Eq. (10) designates the parametric region where the CPA laser is able to appear. The exact CPA laser then requires $N = (2n + 1)\pi/2\phi$ with arbitrary $n$, and if satisfied it leads to infinite transmittance. In many realistic applications, beyond a sufficiently high amplification of the wave field, nonlinear effects become important and the linear theory is no longer valid. It is thus both important and practical to propose structures able to nearly perfectly absorb and efficiently amplify incoming waves but with a finite rate.

Here we stress the fact that a single cell satisfying Eq. (10) allows one to further vary the number of cells and obtain huge amplification and/or absorption. For $\gamma = 2.6$ used in the previous examples, Fig. 3(a) depicts the maximum of transmittance as a function of the number of cells. We observe that the maximum transmission varies significantly even with a small change in the number of cells. The bottom part of the curve in Fig. 3(a) appears to saturate for large lattices. Different peaks, within the range of $N$ plotted here, illustrate an amplification of the wave up to $10^6$ times.

Three different examples of the transmission are shown in Figs. 3(b), 3(c), and 3(d). In Fig. 3(b), we have chosen $N = 14$ since it appears to have a huge maximum transmission, even with only a few unit cells. In contrast, just by adding 1 cell, the transmission in Fig. 3(c) has a maximum

![FIG. 2. (a) The transmittance $T_1$ (dashed) and the envelope function $T_{env}$ (solid) around the $\mathcal{PT}$-broken parametric region where $T_1 > 1$ for $\gamma = 2.6$. (b) The imaginary part of the unit cell transmission coefficient $t$ in the same region.](243904-3)
less than 10. Using panel 3(a), we may choose the maximum possible transmission in this range of cells, which appears at \( N = 590 \), and the corresponding \( T_N \) is shown in Fig. 3(d), reaching an amplification of \( 10^6 \).

The unit cell crosses from the unbroken to the broken phase through a TR with \( T_1 = 1 \) which is also a TR of the finite periodic structure. Two such TRs are indicated with (red) circles in Fig. 2(a). According to the modified conservation law for \( PT \)-symmetric systems [see below Eq. (4)], here there is a totally asymmetric reflection since either \( R_1^{(1)} \) or \( R_0^{(1)} \) is zero. This transmission resonance was discussed in Ref. [40] and corresponds to an exceptional point of the scattering matrix of the unit cell. According to Eq. (6), the reflections from the finite lattice \( R_L^{(N)} \) are analogous to the ones of the unit cell, and these ATRs also appear for the periodic structure.

**Electromagnetic gratings.—**Now, we apply the aforementioned results in a system that has been extensively studied in the context of \( PT \) symmetry and periodicity, i.e., a lattice of electromagnetic gratings or slabs. Here the transverse electric field satisfies Eq. (2) with a refractive index \( \gamma \), where we consider a piecewise constant refractive index in each slab of length \( l/4 \). For this distribution of the refractive index, it is known [29–31] that the \( PT \)-broken phase appears for \( \gamma \geq 1 \).

Since for the unbroken phase the transmission properties are similar to the conservative case, we directly focus on the \( PT \)-broken phase with \( \gamma > 1 \). In particular, choosing \( n_1 = 0.02 \) and \( \gamma = 1.1 \) we identify the region where Eq. (10) is satisfied. The CPA laser for this setting is found close to the Bragg wave number as it is expected [15,29,42]. Figure 4(a) shows the logarithm of the maximum transmission \( T_N \) as a function of the number of cells \( N \), where we observe a more structured pattern with respect to the one of Fig. 3. However, the minimum of \( \log (\max T_N) \) appears to saturate to a limiting value for large \( N \). Its is again illustrated that by satisfying Eq. (10), and changing the number of cells, configurations with huge amplification and/or absorption are found. Two examples of the transmission are shown in Figs. 4(b) and 4(c), illustrating how the number of cells can be used in order to obtain large amplification and/or absorption.

To describe quantitatively both amplification and absorption for the case of the electromagnetic slabs, we perform a singular value decomposition of the scattering matrix from \( N \) cells. The singular values \( \sigma_{\pm} \) are both real and non-negative, and are connected through \( \sigma_{\pm} = 1 \) for a 1D \( PT \)-symmetric system [42].

For a given incoming wave of the form \( \vec{X} = [\psi^+, \psi^-]^T \), where \( \psi^\pm \) are the forward and backward propagating waves, the output to input ratio is \( \Theta = || SX || / || X || \). Then the two singular values correspond to \( \sigma_- = \min || \Theta || \) and \( \sigma_+ = \max || \Theta || \), and quantify the ability of the scatterer to absorb or amplify an incoming wave \( \vec{X} \). At the CPA-laser point, the singular values become \( \sigma_- = 0 \) and \( \sigma_+ \rightarrow \infty \). Here, to illustrate the absorptive and amplifying properties of the system, in Fig. 4(d) we plot \( \sigma_{\pm} \) as a function of \( N \). It is found that by satisfying the conditions of Eq. (10) and varying the number of cells, we obtain a high contrast of absorption and amplification at the same frequency. The dependence on the number of cells is also very sharp in this case.

In summary, we studied finite periodic \( PT \)-symmetric scatterers in one dimension. We obtained a closed-form expression for the transmission from a finite system of \( N \).
cells as a function of the single cell transmission, the Bloch phase, and $N$. A simple envelope function independent of the number of cells was obtained, describing the minima (in the $PT$-unbroken phase) or the maxima (in the $PT$-unbroken phase) of the total transmission. Using this function we find the necessary conditions for a CPA laser, for an arbitrary 1D periodic 1D system, depending only on the unit cell. Although the exact CPA-laser point depends on its total length, here we show that when the necessary conditions obtained for the unit cell are met by varying the number of cells, one can achieve huge amplification and high absorption at the same frequency.