An improved multimodal method for sound propagation in nonuniform lined ducts

WenPing Bi, a Vincent Pagneux, Denis Lafarge, and Yves Aurégan

Laboratoire d’Acoustique de l’Université du Maine, UMR CNRS 6613, Av. O Messiaen, 72085 Le Mans Cedex 9, France

(Received 6 December 2006; revised 6 April 2007; accepted 13 April 2007)

An efficient method is proposed for modeling time harmonic acoustic propagation in a nonuniform lined duct without flow. The lining impedance is axially segmented uniform, but varies circumferentially. The sound pressure is expanded in term of rigid duct modes and an additional function that carries the information about the impedance boundary. The rigid duct modes and the additional function are known \textit{a priori} so that calculations of the true liner modes, which are difficult, are avoided. By matching the pressure and axial velocity at the interface between different uniform segments, scattering matrices are obtained for each individual segment; these are then combined to construct a global scattering matrix for multiple segments. The present method is an improvement of the multimodal propagation method, developed in a previous paper [Bi et al., J. Sound Vib. 289, 1091–1111 (2006)]. The radial rate of convergence is improved from $O(n^{-2})$, where $n$ is the radial mode indices, to $O(n^{-4})$. It is numerically shown that using the present method, acoustic propagation in the nonuniform lined intake of an aeroengine can be calculated by a personal computer for dimensionless frequency $K$ up to 80, approaching the third blade passing frequency of turbofan noise. © 2007 Acoustical Society of America. [DOI: 10.1121/1.2736785]

PACS number(s): 43.50.Gf, 43.20.Mv, 43.20.Fn, 43.20.Hq [LLT]

Pages: 280–290

I. INTRODUCTION

Numerous methods have been proposed to study sound propagation in ducts with locally reactive liners which are mathematically represented by an impedance boundary condition. In the presence of circumferential variations of the lining impedance (e.g., hard walled splices in lined intakes of aeroengine), the problem to solve is fully three dimensional. When dimensionless frequency $K$ is high, where $K=kr$, $k=2\pi f/c$, $f$ is frequency, $c$ is sound velocity in air, and $R$ is the radius of duct, it turns out to be challenging to model it efficiently.

The finite element method (FEM), \textsuperscript{1,2} perturbation method, \textsuperscript{3} and point matching method \textsuperscript{4} have all been proposed to model the sound propagation in nonuniform lined ducts. An equivalent surface source method \textsuperscript{5} and kinetic theory \textsuperscript{6} were also employed to analyze the acoustic field in ducts lined with complicated distributions of impedance.

The hybrid analytical/numerical methods are always interesting, in which analysis is taken as far as possible. One of the hybrid analytical/numerical methods may be the mode matching method. Modes are first calculated in segmented uniform lined ducts and then matched between different uniform segments. When the lining impedance is circumferentially nonuniform, the sound pressure and particle velocity field cannot be separated in the $r-\theta$ plane, the dispersion relation in one segment cannot be written explicitly, and it is not possible to use classical root finding routines to determine the eigenmodes. Watson\textsuperscript{7} used a hard walled duct mode expansion series to numerically evaluate the eigenmodes without flow. The Galerkin method was employed to force the series to satisfy the true boundary condition in the lined segment. The complex modal output amplitudes for a specified source distribution were then obtained by applying a mode matching technique at the discontinuity between rigid and lined segments. Fuller\textsuperscript{8,9} expanded the circumferential admittance function as a Fourier series and the eigenmodes over the separable components adapted to the cylindrical coordinates. He obtained the eigenvalue set without flow to be solved. The axial wave numbers in the lined segment were then calculated by solving this set of equations using the method of Muller. Campos \textit{et al.}\textsuperscript{10} studied acoustic modes in a cylindrical duct with an arbitrary wall impedance distribution with flow. When the wall impedance varies along the circumference, they calculate the acoustic modes in a similar way as Fuller.\textsuperscript{8} Wright\textsuperscript{11} used FEM to calculate the modes and then analytically matched them to the rigid duct modes. Wright \textit{et al.}\textsuperscript{12} also extended it to include uniform flow. Astley \textit{et al.}\textsuperscript{13} proposed a finite element mode matching method for propagation in lined ducts with flow in which the modes are calculated by FEM and numerically matched by a modified Galerkin method.

In Ref. 14, we proposed a multimodal propagation method (MPM) to study sound propagation in a nonuniform lined duct without flow. The sound pressure is expressed as a double series of the rigid duct modes which are known \textit{a priori}. By matching the pressure and axial velocity at the interface between different segments, scattering matrices are obtained for each individual segment and then combined to construct a global scattering matrix for multiple segments. Different kinds of sources can be easily integrated without recalculating the scattering matrix. The full three-dimensional (3D) problem is reduced to a two-dimensional

\textsuperscript{a}Electronic mail: wenping.bi@univ-lemans.fr
one which is better suited for the study of acoustic propagation in the lined intake of an aeroengine. The boundary condition is satisfied in the integral sense. Unlike the methods cited earlier, calculations of the eigenmodes of nonuniform lined ducts, which are very difficult, are avoided. This method is also extended to include uniform flow. Because the individual rigid duct mode does not satisfy the impedance boundary condition, the radial convergence rate is only $O(n^{-2})$, where $n$ is the radial mode indices.

In this paper, we improve the MPM by accelerating the radial convergence rate. The sound pressure is expressed as a double series of the rigid duct modes and an additional function which carries the information of the impedance boundary. This is motivated by Ref. 17 which studies water waves over variable bathymetry regions, and Refs. 18 and 19 which study sound propagation in rigid waveguides with varying cross section. It is shown that the radial convergence rate of the double infinite series is improved from $O(n^{-2})$ to $O(n^{-4})$. This improvement can be extended to include uniform flow.

The paper is organized as follows. In Sec. II, the derivation of equations are presented for circumferentially uniform and nonuniform boundary conditions, respectively. The convergence properties are then shown analytically and numerically in Sec. III. Finally, in Sec. IV, we present some numerical examples to show the robustness and capability of the method.

II. DERIVATION OF THE MULTIMODAL EQUATIONS

We consider an infinite rigid duct with circular cross section lined with a region of nonuniform liner. The liner properties are assumed to be given by a distribution of locally reacting impedance. Without significant loss of generality, the distribution may be assumed axially segmented, i.e., the impedance is set piecewise constant along the duct, while being arbitrarily variable along the circumference of each segment. In Fig. 1 the configuration of one axial segment of lining impedance is depicted, the circumferential variation of impedance is presented as two acoustically rigid splices, which is a typical configuration in the intake of an aeroengine. Linear and lossless sound propagation in air is assumed. With time dependence $e^{i\omega t}$ omitted, the equation of mass conservation combined with the equation of state, and the equation of momentum conservation are written as

$$\nabla \cdot v = -\frac{j\omega}{p_0 c_0} p,$$

$$j\omega v = -\frac{1}{\rho_0} \nabla p,$$

where $v$ is the particle velocity, $p$ is the acoustic pressure, and $\rho_0$ and $c_0$ are the ambient density and speed of sound in air. Pressures, velocities, and lengths are, respectively, divided by $\rho_0 c_0^2$, $c_0$ and $R$ (the duct radius) to reduce Eqs. (1) and (2) to the dimensionless form

$$\nabla \cdot v = - jK p,$$

$$-jKv = \nabla p,$$

where $K = \omega R / c_0$ is the dimensionless wave number. This yields the 3D wave equation

$$\nabla^2 p + \frac{\partial^2 p}{\partial z^2} + K^2 p = 0,$$

where

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

The radial boundary condition is
\[ \frac{\partial p}{\partial r} = Y(\theta) p \quad \text{at } r = 1, \]  
(7)

where \( Y(\theta) = -jK\beta_\theta \), and \( \beta_\theta \) is the liner admittance.

For the sake of clarity, we first consider a problem with a circumferentially uniform boundary condition. After this “warm up,” the circumferentially nonuniform problem is investigated.

A. Circumferentially uniform impedance boundary condition

When the lining impedance is circumferentially uniform, the boundary condition is written as

\[ \frac{\partial p}{\partial r} = Y_0 p \quad \text{at } r = 1, \]  
(8)

where \( Y_0 = -jK\beta_0 \) and \( \beta_0 \) is the liner admittance and it is a complex constant.

In contrast to Ref. 14, the solution of Eq. (5) with boundary condition (8) is expressed as an infinite series and an additional function in order to satisfy the boundary condition

\[ p(r, \theta, z) = \sum_{n=0}^{\infty} P_{mn}(z)\Psi_{mn}(r, \theta) + A_m(z)\chi_m(r)e^{-j\text{m}\theta}, \]  
(9)

where \( P_{mn} \) are the expansion coefficients and \( m \) and \( n \) refer to azimuthal and radial mode indices, respectively. It is noted that because the lining impedance is circumferentially uniform, there is no coupling between azimuthal modes \( m \), Eq. (9) involves only the coupling between radial modes \( n \). The basis functions

\[ \Psi_{mn} = \frac{1}{\sqrt{\pi\Lambda_{mn}}} J_m(\alpha_{mn}r)e^{-j\text{m}\theta} \]  
(10)

are the eigenfunctions of the hard walled cylindrical circular duct which obey the transverse Laplacian eigenproblem

\[ \left[ \frac{1}{r}\frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) - \frac{m^2}{r^2} \right] \Psi_{mn} = -\alpha_{mn}^2 \Psi_{mn}, \]  
(11)

with hard walled boundary condition

\[ \frac{\partial \Psi_{mn}}{\partial r} = 0 \quad \text{at } r = 1, \]  
(12)

and the orthogonality relation

\[ \int \Psi_{mn}(r, \theta)\Psi_{m'n'}(r, \theta)dS = \delta_{m,m'}\delta_{n,n'}, \]  
(13)

where the asterisk denotes the complex conjugate and \( \delta \) denotes the Kronecker delta. The normalization coefficients \( \Lambda_{mn} \) are as follows:

\[ \Lambda_{mn} = 1 - \frac{m^2}{\alpha_{mn}^2}. \]  
(14)

In order to choose \( A_m\chi_m \), Eq. (9) is substituted into the boundary condition (8) and using Eq. (12),

\[ A_m(z)\frac{\partial \chi_m(r)}{\partial r} e^{-j\text{m}\theta} = Y_0 \left[ \sum_{n=0}^{\infty} P_{mn}(z)\Psi_{mn}(r, \theta) \right. \]  
\[ + A_m(z)\chi_m(r)e^{-j\text{m}\theta} \]  
\[ \left. \right] \quad \text{at } r = 1. \]  
(15)

Function \( \chi_m(r) \) can be chosen freely. From Eq. (15), it is shown that for deciding \( A_m(z) \), two conditions can be imposed on \( \chi_m(r) \),

\[ \chi_m(r)|_{r=1} = 0, \]  
(16a)

\[ \frac{d\chi_m(r)}{dr} \bigg|_{r=1} = 1. \]  
(16b)

Substitution of the conditions (16) into Eq. (15) yields

\[ A_m(z) = Y_0 \sum_{n=0}^{\infty} P_{mn}(z)\Psi_{mn}(1, \theta)e^{j\text{m}\theta} = Y_0 \sum_{n=0}^{\infty} \frac{P_{mn}(z)}{\sqrt{\pi\Lambda_{mn}}}. \]  
(17)

Functions which satisfy the conditions (16) may be not unique. One choice may be

\[ \chi_m(r) = B_m J_m(\beta_{m,0}r), \]  
(18)

where \( B_m \) is constant, \( J_m \) is the \( m \) order first kind Bessel function, and \( \beta_{m,0} \) refers to the roots of

\[ J_m(\beta_{m,0}) = 0. \]  
(19)

To obtain the constant \( B_m \), substitution of Eq. (18) into Eq. (16b) yields

\[ B_m = \frac{-1}{\beta_{m,0} J_{m+1}(\beta_{m,0})}. \]  
(20)

Substitution of Eqs. (17), (18), and (20) into Eq. (9), yields

\[ p(r, \theta, z) = \sum_{n=0}^{\infty} P_{mn}(z)\Psi_{mn}(r, \theta) \]  
\[ + \sum_{n=0}^{\infty} P_{mn}(z) - Y_0 J_m(\beta_{m,0}r)e^{-j\text{m}\theta} \]  
\[ \times \frac{1}{\sqrt{\pi\Lambda_{mn}}} \beta_{m,0} J_{m+1}(\beta_{m,0}) \]  
\[ \times \frac{1}{\sqrt{\pi\Lambda_{mn}}} \beta_{m,0} J_{m+1}(\beta_{m,0}). \]  
(21)

For calculating \( P_{mn}(z) \), we project \( p(r, \theta, z) \) on the basis \( \Psi_{mn} \). Following the matricial terminology, Eq. (9) is written

\[ p(r, \theta, z) = \Psi^T M P, \]  
(22)

where \( P \) and \( \Psi \) are column vectors, the superscript “\( T \)” indicates the transpose. \( M \) is a matrix, it is equal to

\[ M = I + 2\pi Y_0 N \Phi \Phi^T, \]  
(23)

where \( I \) refers to the identity matrix, \( N \) is a diagonal matrix, its elements in the main diagonal are \( 1/(\alpha_{mn}^2 - \beta_{m,0}^2) \). They come from the projection of \( \chi_m \) over the rigid mode eigenfunctions as shown in Appendix A. \( \Phi \) is a column vector, its elements are \( 1/\sqrt{\pi\Lambda_{mn}} \).

Using Eqs. (22) and (23), we project Eq. (5) to yield

\[ MP' + AP = 0, \]  
(24)

where matrix \( A \) is
\[ A = (K^2 I - L) M + 2 \pi Y_0 \Psi'(1, \theta) \Psi^T(1, \theta), \]  \hspace{1cm} (25) 

L is a diagonal matrix, its elements in the main diagonal are \( \alpha_{2n}^2 \), and the double prime refers to the second derivative with respect to axial coordinate \( z \). The projection of \( \nabla_z^2 p \) of Eq. (5) is shown in Appendix A.

It is noted that because the boundary condition is circumferentially uniform, there is no coupling between azimuthal modes. The indices \( m \) and \( n \) of the above-mentioned vectors and matrices are \( m = m_0 \) and \( 0 \leq n < \infty \).

### B. Circumferentially nonuniform impedance boundary condition

When the boundary condition is circumferentially nonuniform as in Eq. (7), we have to solve a full 3D problem. The sound pressure cannot be separated in the \( r - \theta \) plane. We do not succeed in finding a function to exactly satisfy the boundary condition (7). On the other hand, a function is found to satisfy the nonuniform boundary condition in the sense of \( \partial p / \partial r = (\sum Y_m e^{-j m \theta}) p \), where \( Y_m \) is the Fourier transformation coefficients of \( Y(\theta) \). Similar to Eq. (9), the sound pressure is expressed as

\[ p(r, \theta, z) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} P_{mn}(z) \Psi_{mn}(r, \theta) \]

\[ + \sum_{m=-\infty}^{\infty} A_m(z) \chi_m(r) e^{-j m \theta}. \]  \hspace{1cm} (26)

As in Sec. II A, substitution of Eq. (26) into the boundary condition (7) yields

\[ A_m(z) = \frac{1}{2 \pi} \int_0^{2\pi} Y(\theta) e^{j m \theta} d\theta \sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} P_{m'n'}(z) \Psi_{m'n'}(1, \theta) \]

\[ = \sum_{m'=0}^{\infty} \frac{1}{2 \pi} \int_0^{2\pi} Y(\theta) e^{-j m' \theta} d\theta \sum_{n'=0}^{\infty} \frac{P_{m'n'}(z)}{\sqrt{\pi A_{m'n'}}}. \]  \hspace{1cm} (27)

where we have used Eq. (10) and imposed the conditions (16). Function \( \chi_m \), satisfying conditions (16), is the same as in Eqs. (18) and (20).

Following the matricial terminology, Eq. (26) is written

\[ p(r, \theta, z) = \Psi^T \Phi P \]  \hspace{1cm} (28)

where \( \Phi \) is

\[ \Phi = I + NY^* \Phi^T. \]  \hspace{1cm} (29)

\( N \) and \( \Phi \) are the same as in Eq. (23), respectively, and the elements of the matrix \( Y \) are \( f_0^{2\pi} Y(\theta) e^{-j (m'-m) \theta} d\theta \).

Using Eqs. (28) and (29), we project Eq. (5) to yield

\[ MP'' + AP = 0, \]  \hspace{1cm} (30)

where matrix \( A \) is

\[ A = (K^2 I - L) M + \int_0^{2\pi} Y(\theta) \Psi^*(1, \theta) \Psi^T(1, \theta) d\theta, \]  \hspace{1cm} (31)

where \( L \) is the same as in Eq. (25), a diagonal matrix, its elements in the main diagonal are \( \alpha_{2n}^2 \), the double prime refers to the second derivative with respect to axial coordinate \( z \). The projection of \( \nabla_z^2 p \) of Eq. (5) is shown in Appendix A.

It is noted that because the boundary condition is circumferentially nonuniform, modes are coupled between azimuthal orders. The indices \( m \) and \( n \) of the above-mentioned vectors and matrices are \( -\infty < m < \infty \) and \( 0 \leq n < \infty \).

Equations (24) and (30) are constant coefficient matrix differential equations when the axial lining impedance is uniform in one segment. Their solutions can be directly written as

\[ P = XD(z) C_1 + XD(l - z) C_2. \]  \hspace{1cm} (32)

where \( C_1 \) and \( C_2 \) are amplitude vectors of dimension \( N \times \), \( (M \times N \) where \( M \) and \( N \) refer to the truncated dimensions of mode indices \( m \) and \( n \), \( X \) is the \( N \times N \) matrix whose columns are the generalized eigenvectors \( X_n \) of matrix \( M^{-1} A \), and \( D(z) \) and \( D(l - z) \) are diagonal matrices with \( \exp(-j v_n z) \) and \( \exp(-j v_n(l - z)) \), respectively, on the main diagonal, with \( v_n = \sqrt{\alpha_n} \), \( \alpha_n \) being the generalized eigenvalues of matrix \( M^{-1} A \). In the form of Eq. (32), numerical stability is ensured because the propagation matrices \( D(z) \) and \( D(l - z) \) have only positive arguments and contain no exponentially diverging terms due to the evanescent modes. By matching the pressure and axial velocity at the interfaces of the segment, the coefficients of transmission and reflection are yielded. Scattering matrices are then obtained for each individual segment; these are combined to construct a global scattering matrix for multiple segments. This procedure is the same as in Ref. 14 and outlined in detail in Appendix B.

III. CONVERGENCE ANALYSIS

In Ref. 14, the sound pressure is expressed as a double series of the rigid duct modes, which are known \textit{a priori}. It is numerically shown that the convergence rates are \( O(n^{-2}) \) when \( m \) is fixed, and \( O(m^{-3}) \) when \( n \) is fixed. The convergence rate for \( n \) is slow because the rigid duct modes do not satisfy individually the impedance boundary condition. This slow convergence rate is improved in this paper by adding a function which satisfies the boundary condition. In this section, the behaviors of the convergence rate are shown analytically and numerically.

In general, the MPM\textsuperscript{14} and the method presented in this paper are the generalized Fourier series method. Their convergence rate can be estimated by the divergence theorem or integration by parts. Let us take a function \( g \) on the segment [0,1] with \( g'(0)=0 \) and \( g'(1)=a \) and with its second derivative \( g'' \) integrable. If we project this function on the Neumann basis \( u_n(x)=\sqrt{2-\delta_{n0}} \cos(n \pi x) \) with \( u'(0)=u'(1)=0 \), then \( g(x)=\sum_{n=0}^{\infty} G_n u_n(x) \) and by integration by parts

\[ G_n = \int_0^1 g u_n dx = -\frac{1}{n^2 \pi^2} \left[ -au_n(1) + \int_0^1 g'' u_n dx \right]. \]  \hspace{1cm} (33)
Since $g''$ is integrable, by the Riemann-Lebesgue lemma we know that \( \lim_{r \to -\infty} \int_{0}^{2\pi} g'' u_r \, dx = 0 \). Consequently

\[
G_n = \frac{\sqrt{2}a(-1)^{n+1}}{\pi n^2} + o\left(\frac{1}{n^2}\right),
\]

so that the leading term for $G_n$ is given by the derivative of $g$ at the boundary.

Consider sound pressure $p(r, \theta, z)$ to satisfy Helmholtz equation (5) with impedance boundary condition (7) in an infinite lined duct. Because we are interested in the radial convergence rate, without loss of generality, the axial lining impedance can be assumed as uniform. In the following, we assume also that the $p(r, \theta, z)$ is sufficient differential. Equations (9) and (26) can be expressed in a generalized Fourier series

\[
p(r, \theta, z) = \sum_i P_i(z) \Phi_i(r, \theta),
\]

where $\Phi_i$ is any set of functions which are complete and orthogonal. If the basis functions $\Phi_i$ satisfy the transverse Laplace eigenproblem

\[
\nabla^2 \Phi_i = -\gamma_i^2 \Phi_i,
\]

using the divergence theorem, the expansion coefficients $P_i$ can be written as

\[
P_i = \int p \Phi_i dS = \frac{-1}{\gamma_i^2} \int \nabla^2 p \Phi_i dS + \frac{1}{\gamma_i^2} \times \oint \left( \Phi_i \frac{\partial p}{\partial r} - p \frac{\partial \Phi_i}{\partial r} \right) dC,
\]

where $\gamma_i$ are the transverse Helmholtz wave numbers corresponding to $\Phi_i$. When the basis functions $\Phi_i$ satisfy the boundary condition

\[
\frac{\partial \Phi_i}{\partial r} = Y(\theta) \Phi_i \quad \text{at} \quad r = 1,
\]

individually, the second term in Eq. (37) is equal to zero. This is the so-called eigenfunction expansion, i.e., $\Phi_i$ and $\gamma_i$ are the eigenfunctions and eigenvalues of the Helmholtz equation with corresponding impedance boundary conditions in the nonuniform lined ducts. Repeated use of the divergence theorem and Helmholtz equation (5) leads to an exponential convergence rate of $P_i$.

When the basis functions $\Phi_i$ are not the eigenfunctions of lined ducts, e.g., $\Phi_i = \Psi_{mn}$, where $\Psi_{mn}$ are the rigid duct eigenfunctions, which do not satisfy the boundary condition (38) individually, the convergence rate will be slow. In this case, the expansion coefficients $P_{mn}$ are

\[
P_{mn} = \frac{-1}{\alpha_{mn}^2} \int \nabla^2 p \Psi^*_{mn} dS + \frac{1}{\alpha_{mn}^2} \times \oint \left( \Psi^*_{mn} \frac{\partial p}{\partial r} - p \frac{\partial \Psi^*_{mn}}{\partial r} \right) dC
\]

\[
= \frac{-1}{\alpha_{mn}^2} \int \nabla^2 p \Psi^*_{mn} dS + \frac{1}{\alpha_{mn}^2} \sum \frac{P_{m'n'}}{\pi \sqrt{\Lambda_{m'n'} \Lambda_{mn}}} \int_0^{2\pi} Y(\theta)e^{-j(m'-m)\theta} d\theta,
\]

where $\alpha_{mn}$ are the eigenvalues of a rigid duct as mentioned earlier. The divergence theorem can be used a second time to show that the first term is $o(1/\alpha_{mn}^2)$. For the second term, it is evident that at fixed $m$, $\sum_{m'n'} P_{m'n'}/ (\pi \sqrt{\Lambda_{m'n'} \Lambda_{mn}}) \int_0^{2\pi} Y(\theta)e^{-j(m'-m)\theta} d\theta$ does not affect the radial convergence rate of $P_{mn}$. Hence, at fixed $m$, Eq. (39) yields

\[
P_{mn} = O\left(\frac{1}{\alpha_{mn}^2}\right).
\]

The asymptotic forms of $\alpha_{mn}$ at fixed $m$ and $n > N_0$, where $N_0$ is a sufficiently large constant, is

\[
\alpha_{mn} = \left(n + \frac{m - 3}{4}\right) \pi + O\left(\frac{1}{n}\right).
\]

We therefore obtain the rates of convergence of $P_{mn}$ with respect to index $n$ at fixed $m$,

\[
P_{mn} = O\left(\frac{1}{n\alpha_{mn}^2}\right), \quad n > N_0.
\]

For the method presented in this paper, an additional function is involved in the expression of $p(r, \theta, z)$, the $P_{mn}$ is

\[
P_{mn} \sim \int \rho \Psi^*_{mn} dS \sim \int \sum_{m'=-\infty}^\infty A_{m'} \chi_{m'} e^{-j\theta \text{m'}m} \Psi^*_{mn} dS
\]

\[
= \frac{-1}{\alpha_{mn}^2} \int \nabla^2 p \Psi^*_{mn} dS + \frac{1}{\alpha_{mn}^2} \oint \left( \Psi^*_{mn} \frac{\partial p}{\partial r} - p \frac{\partial \Psi^*_{mn}}{\partial r} \right) dC
\]

\[
= \frac{-1}{\alpha_{mn}^2} \sum \frac{P_{m'n'}}{\pi \sqrt{\Lambda_{m'n'} \Lambda_{mn}}} \int_0^{2\pi} Y(\theta)e^{-j(m'-m)\theta} d\theta,
\]

where we have used Eq. (A1). Using the divergence theorem a second time for the first term we find that the first term is at least $O(1/\alpha_{mn}^4)$. Equation (43) is written as

\[
P_{mn} = O\left(\frac{1}{\alpha_{mn}^2}\right) + \sum \frac{P_{m'n'}}{\pi \sqrt{\Lambda_{m'n'} \Lambda_{mn}}} \int_0^{2\pi} Y(\theta)e^{-j(m'-m)\theta} d\theta - \frac{1}{\alpha_{mn}^2}
\]

\[
\times \int_0^{2\pi} Y(\theta)e^{-j(m'-m)\theta} d\theta \times \frac{1 + \beta_{m,0}^2}{\alpha_{mn}^2} + O\left(\frac{\beta_{m,0}^2}{\alpha_{mn}^4}\right)
\]

\[
\times \sum \frac{P_{m'n'}}{\pi \sqrt{\Lambda_{m'n'} \Lambda_{mn}}} \int_0^{2\pi} Y(\theta)e^{-j(m'-m)\theta} d\theta
\]

\[
= O\left(\frac{1}{\alpha_{mn}^2}\right).
\]
The radial convergence rate of the present method is $O(n^{-4})$ as $n > N_0$, where $N_0$ is a sufficiently large constant, when $m$ is fixed.

(2) $N_0$ depends on the azimuthal order $m$. For a small $m$, $N_0$ is small, $P_{mn}$ converges as $O(n^{-3})$ after a few terms of $n$. For a large $m$, however, $P_{mn}$ converges as $O(n^{-4})$ after a large $n$.

(3) It is important to note that the behavior of convergence rate Eq. (45) is independent of the variation of admittance $Y(\theta)$. It means that whether $Y$ is constant, i.e., no circumferential mode scattering, or $Y$ varies circumferentially, i.e., there is circumferential mode scattering, the convergence behavior does not change for fixed $m$.

Now, the radial convergence properties are numerically shown in Figs. 2–4 for fixed $m$. The configuration is the same as in Fig. 1. An infinite rigid duct is lined with one axial segment impedance with two acoustically rigid splices distributed oppositely. The splice angles are 0.06 rad, dimensionless frequency is $K=31.26$, lining impedance is $Z/\rho c$

\[
+ O\left(\frac{\beta_{m0}^2}{\alpha_{mn}^4}\right) \sum_{m'n'} \frac{P_{m'n'}}{\pi\Lambda_{m'n'}\Lambda_{mn}} \int_0^{2\pi} Y(\theta) e^{-j(m'-m)\theta} d\theta. \tag{44}
\]

When $m$ is fixed, using Eq. (41), Eq. (44) is written as

\[
P_{mn} = O\left(\frac{1}{n + \frac{m}{2}}\right), \quad n > N_0. \tag{45}
\]

This convergence rate is valid for both circumferentially uniform and circumferentially nonuniform boundary conditions. From Eq. (45), it is shown that

(1) The radial convergence rate of the present method is $O(n^{-4})$ as $n > N_0$, where $N_0$ is a sufficiently large constant, when $m$ is fixed.

(2) $N_0$ depends on the azimuthal order $m$. For a small $m$, $N_0$ is small, $P_{mn}$ converges as $O(n^{-3})$ after a few terms of $n$. For a large $m$, however, $P_{mn}$ converges as $O(n^{-4})$ after a large $n$.

(3) It is important to note that the behavior of convergence rate Eq. (45) is independent of the variation of admittance $Y(\theta)$. It means that whether $Y$ is constant, i.e., no circumferential mode scattering, or $Y$ varies circumferentially, i.e., there is circumferential mode scattering, the convergence behavior does not change for fixed $m$.
IV. NUMERICAL EXAMPLES

In this section, some examples corresponding to the problem of Fig. 1 are presented to show the capability of the method. In Fig. 6, \( \frac{\mathrm{abs}(W_N^o-W_{1000}^o)}{W_{1000}^o} \) versus different truncations \( N \) is shown, where \( W_N^o \) refers to the output sound power at the exit plane (Fig. 1) for truncation \( N \). \( W_{1000}^o \) is the converged value. The lining admittance is circumferentially uniform, i.e., \( Y(\theta)=Y_0 \). The parameters are noted in Fig. 6. Incident amplitudes are equal to 1. For an incident mode \((m=2, n=0)\) and with the truncation \( N=10 \), relative error 1% is obtained compared with truncation \( N=1000 \). For incident mode \((m=98, n=0)\), when the truncation \( N=25 \), relative error 1% is obtained compared with truncation \( N=1000 \). Another example is shown in Fig. 7. The lining impedance is 
\[
Z/c = 2 - j0.01K/R - \cot(0.016K/R),
\]
where \( K=10, R=0.2 \). When mode \((m=0, n=0)\) is incident, surface modes are invoked. Such surface modes are located near the duct wall and exponentially decay away from the duct wall. This is
the most difficult case for the MPM\textsuperscript{14} and the improved method presented in this paper. For MPM,\textsuperscript{14} only rigid duct modes are used to express this surface mode, about $N=200$ rigid duct modes are needed to converge to the surface mode. However, only $N=30$ terms are needed for the improved method in this paper to converge to the surface mode.

A full 3D example with high dimensional frequency $K$ is shown in Figs. 8 and 9. The configuration is the same as in Fig. 1. An infinite rigid duct is lined with one axial segment impedance with two rigid splices distributed oppositely. The splice angles are 0.06 rad. The dimensionless frequency is $K=80$, the lining impedance is $Z/\rho c=2+j$. The parameters are typical of aeroengine intakes. The dimensionless frequency $K=80$ approximately corresponds to the third BPF of turbofan noise. However they do not exactly correspond to any true aeroengine. Due to the lined segment, the incident mode is scattered to different azimuthal and radial modes. The sound power of every rigid mode at the exit plane is

![Graph](image1)

**FIG. 5.** Convergence rates of radial order $n$ for different incident modes $(m_0, 0)$. Solid line refers to $m_0=98$, solid line with squares refers to $m_0 =2$, dashed line with stars refers to $n^{-4}$, dashed line with circles refers to $n^{-3}$, and dashed line with plus refers to $n^{-2}$. $K=31.26$, $Z/\rho c=2+j$, no splice.

![Graph](image2)

**FIG. 6.** $|W_0 - W_{1000}|/W_{1000}$ vs different truncation $N$. Solid line refers to incident mode $(m=2, n=0)$, dashed line refers to incident mode $(m=98, n=0)$. $K=110$, $Z/\rho c=2+j$, $L/R=0.48$, no splice.
shown in the figures. This example may be difficult for purely numerical methods, e.g., FEM. For the present method, it takes hours on a personal computer (about 2 h on a PC with processor P IV 2.4 GHz, 512M physical RAM and 3G virtual memory, without optimizing the MATLAB code). In Fig. 8, mode \((m=2, n=0)\) is incident. The scattered modes are distributed nearly symmetrically for \(+m\) and \(-m\). In Fig. 9, mode \((m=76, n=0)\), the last propagating mode, is incident. The incident mode is scattered to all modes with even \(m\) that have lower azimuthal orders. The performance of the liner will be reduced by the presence of rigid splices.

**V. CONCLUSIONS**

An efficient method is developed to model acoustic propagation in a nonuniform lined duct. The sound pressure is expressed as a double series of rigid duct modes and an additional function carrying the information about the impedance boundary, which are known *a priori*. Calculations of the eigenmodes of a lined duct, which are difficult, are avoided. The radial convergence rate is accelerated from \(O(n^{-2})\) to \(O(n^{-4})\) for fixed \(m\), where \(n\) and \(m\) are the radial and azimuthal indices, respectively. Numerical examples show that this method can deal with a full 3D problem with high dimensionless frequency \(K\), e.g., \(K=80\) corresponding to the third BPF in an aeroengine.

**APPENDIX A: PROJECTIONS OF \(P\) AND \(\nabla^2 P\)**

Using Eqs. (26), (27), (18), and (20), the projection of \(p\) over the basis \(\Psi^*\) for circumferentially nonuniform boundary condition is

---

FIG. 7. \(W^o_N\) vs \(N\) for different methods. Solid line refers to the improved method, dashed line refers to the MPM. \(K=10, R=0.2, Z/c = j(0.01K/R−\cot(0.016K/R)), L/R =2.5\), mode \((0, 0)\) is incident, \(|P_{0,0}| =1\), no splice. There exist surface waves, which is the most difficult case for the MPM or the improved method.

FIG. 8. Output sound power of rigid modes at the exit section. \(K=80, Z/c =2+j, L/R =0.48\), mode \((m=2, n=0)\) is incident, \(|P_{2,0}| =1\), two splices with angle 0.06 rad. Only modes with even \(m\) are excited and shown.
The projection of $\nabla \cdot \rho \Psi_{mn}^* dS$ is

$$\int \nabla \cdot \rho \Psi_{mn}^* dS = \int p \nabla \cdot \rho \Psi_{mn}^* dS$$

$$+ \oint \left( \Psi_{mn}^* \frac{\partial \rho}{\partial r} - \rho \frac{\partial \Psi_{mn}^*}{\partial r} \right) dC$$

$$= \int p(-\alpha_{mn}^2) \Psi_{mn}^* dS + \oint \Psi_{mn}^*(1, \theta) d\theta dC$$

$$= -\alpha_{mn}^2 \sum_{m'n'} M_{mn,m'n'} P_{m'n'}$$

$$+ \sum_{m'n'} \int_{0}^{2\pi} Y(\theta) \Psi_{mn}'(1, \theta)$$

$$\times \Psi_{m'n'}(1, \theta) d\theta P_{m'n'}.$$

(A3)
APPENDIX B: CALCULATIONS OF T AND R

The reflection and transmission matrices are easily identified by writing the general field solution in the lined section, and the continuity conditions at the interfaces.

Using the solutions (32) in the lining segment and the continuity of pressure and axial velocity leads to

\[ A_1 + B_1 = MX(C_1 + D_1C_2), \]

\[ K_0(A_1 - B_1) = MXK_V(C_1 - D_1C_2), \]

\[ A_2 + B_2 = MX(D_1C_1 + C_2), \]

\[ K_0(A_2 - B_2) = MXK_V(D_1C_1 - C_2), \] (B1)

where \( D_l = D(l), \) \( K_0, \) and \( K_V \) are diagonal matrices with the axial wave numbers on the main diagonal in the rigid and lined sections (respectively, the \( K_{0,mn} \) and \( d_m \)). \( A_1, B_1, A_2, \) and \( B_2 \) are the modal amplitudes in the rigid duct respectively as shown in Fig. 1.

By denoting

\[ F = MX + K_0^{-1}MXK_V, \] (B2)

\[ G = MX - K_0^{-1}MXK_V, \] (B3)

Eq. (B1) is reduced to

\[ 2A_1 = FC_1 + GD_1C_2, \]

\[ 2B_1 = GC_1 + FD_1C_2, \]

\[ 2A_2 = FD_1C_1 + GC_2, \]

\[ 2B_2 = GD_1C_1 + FC_2, \] (B4)

The reflection and transmission matrices, which completely characterize the segmented liner, are then given by

\[ T = t = (FD_l - GF^{-1}GD_l)(F - GD_lF^{-1}GD_l)^{-1}, \]

\[ R = r = (G - FD_lF^{-1}GD_l)(F - GD_lF^{-1}GD_l)^{-1}, \] (B5)

where we have used the definition of \( T, R, \) and scattering matrix \( S \) of a single lining segment

\[
\begin{pmatrix}
A_2 \\
B_1
\end{pmatrix} = S
\begin{pmatrix}
A_1 \\
B_2
\end{pmatrix}
\]

where

\[ S = \begin{bmatrix} T & R \\ R & T \end{bmatrix}. \] (B6)

The global scattering matrix of multiple lining segments are then easily obtained as given in Ref. 14.